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# Comparison of the modelling techniques of dynamical systems using momentum and energy principles

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**Abstract**

In undergraduate courses such as engineering mechanics and modelling and simulation, students mostly use Newton's second law to derive equations of motion in dynamics problems. However, it might be easier and faster in some dynamics problems to use Lagrange's equation instead of Newton's second law. To analyze the reason of this situation we will compare the general features of each type of these methods from different aspects. Then we will apply both to some examples and indicate the differences and levels of difficulty accompanying the application of each technique. Three examples from mechanical systems and one example from electrical systems are presented here to show the basic differences between Newton's second law of motion and Lagrange's equation. Three examples include a system with a single degree of freedom (translational), two degrees of freedom and multiple degrees of freedom in translation and rotation. For electrical systems, a two degrees of freedom circuit with dual portions is explained using Lagrange's equation and Kirchhoff law. The advantages and disadvantages of each approach and how the principal concepts in system dynamics are explained to the students are also highlighted.

**Keywords:** Modelling, Engineering Mechanics, Energy, Newton's Second Law, Lagrange's Equation

## Introduction

The main objective of modelling any dynamical system is to find the equation of motion of the system components to identify the size of the actuators. The equation of motion is a relation between mass, force and acceleration in translational motion or Inertia or torque and angular acceleration in rotational motion. The equation of motion of any dynamical systems is very important in controlling the system using the classical control theory. In the absence of the equation of motion, we have to apply evolutionary algorithms to control the dynamical systems. In modelling, we have two major approaches: Newton's second law of motion based on momentum quantity which is a vector approach and Lagrange's equation based on energy principle. Newton's laws first appeared in his masterpiece, Philosophiae Naturalis Principia Mathematica (1687), commonly known as the Principia [1]. Newton's second law of motion pertains to the behavior of objects for which all existing forces are not balanced. It states that the acceleration of an object is directly proportional to the net force affecting the object and inversely proportional to the mass of this object [1].

From mathematical point of view, Newton's second law is a vector second order ordinary differential equation that can be reduced to more than one scalar second order differential equations. Each scalar ordinary differential equation represents motion equation in certain direction. Two integration steps are needed later to reduce the differential equation to algebraic equation. While Lagrange's equation, which is based on energy principles, involves number of first order partial differential equations depends on the number of required generalized coordinates in a certain application. This could be one of the reasons why fresh engineering students are directed to use Newton's method instead of Lagrange's method.

From physics point of view, when a student applies Newton's second law to derive equation of motion of a dynamical system, he usually feels the physics of the problem. For example, a student should determine the direction of a force to write its correct sign in the equation. Sometimes, this is not an easy task as the nature of the force, either active or resisting, is not clear from the free body diagram. The student needs to guess the possibility of motion direction and the rule of each force. This step cannot be done as separate from understanding the physics of the problem. However, this process becomes more and more difficult when the number of forces affecting one body increases or the problem itself includes more than one body. In this situation, Lagrange's equation invented by Joseph-Louis Lagrange is the best choice for easy and direct way to obtain the equation of motion of the dynamical systems.

Joseph-Louis Lagrange (1736- 1813, Paris, France), Italian French mathematician who made great contributions to number theory and to analytic and celestial mechanics introduced the principle of energy approach in modelling dynamical systems. His most important book, Mécanique analytique (1788; "Analytic Mechanics"), was the basis for all later work in this field. It is almost impossible to include examples from all applications to represent differences between Newton's method and Lagrange's method. We will focus on examples that are used mainly in the undergraduate curricula. However, these examples will range from systems with one degree of freedom to systems with multi degrees of freedom and are of gradual levels of difficulty. It should be noted that most of the selected examples are available already in some textbooks, but the comments are drawn based on the experience of the authors in teaching undergraduate and postgraduate engineering courses.

It is the main objective of this study to compare the famous two mathematical modelling techniques in deriving equations of motion of mechanical dynamical systems: Newton's second law of motion and Lagrange's Equation. Also, the application of Lagrange's equation in electrical systems using system similarity is also investigated. Procedure, merits and difficulties are also highlighted. This paper is organized as follows: Section (1) is the introduction. Section (2) contains four examples, and they are modelled using both momentum and energy. Discussion about advantages and disadvantages, and limitation of each technique are presented in section (3). Section (4) is the concluding remarks followed by references.

## 2. Case Studies

### 2.1 Example 1

consider a system with single degree of freedom in which a mass $m$ slides on inclined smooth surface of angle $\theta$ where spring and dashpot resist its motion as shown in Figure 1.

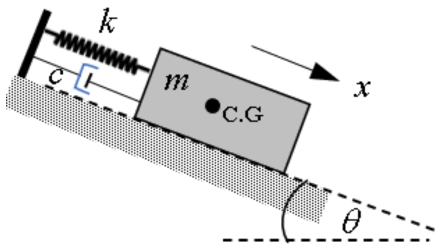



Fig. 1 a system with single degree of freedom

As the mass moves in a straight line on the surface, one equation of motion is required to describe its rectilinear motion.

#### 2.1.1 Newton's method

To derive the equation of motion using Newton's second law a student needs to follow the following steps:

1) Draw the free body diagram of the mass showing all external forces.
2) Determine the direction in which Newton's second law will be applied.
3) Determine acceleration and forces components in this direction.
4) Identify nature of each force whether it is assisting or resisting motion.

The equation of motion in the direction *x* down the surface will be.

$$m\ddot{x} = W\sin\theta - kx - c\dot{x} \tag{1}$$

Where the displacement x is measured from an arbitrary reference. When considering only the displacement from equilibrium position *x* the component of the weight will be canceled out by the initial static spring force. Then equation (1) becomes in its final form

$$m\ddot{x} = -kx - c\dot{x} \tag{2}$$

### 2.1.2 Lagrange's Equation

Lagrange's equation is one of the easiest and most straight forward technique to obtain the equations of motion in any dynamical system. This method can be applied for any number of degrees of freedom either for translation or rotation motion. The method consists of five known steps in a clear and precise manner and the students can find the equation of motion accurately. Once the generalized coordinates are selected, the technique can be applied to find equation of motion related to each generalized coordinate. The Lagrange's Equation is given by [2]:

$$\frac{d}{dt}\left(\frac{\partial K.E.}{\partial \dot{x}_j}\right) - \left(\frac{\partial K.E.}{\partial x_j}\right) + \left(\frac{\partial P.E.}{\partial x_j}\right) + \left(\frac{\partial R}{\partial \dot{x}_j}\right) = Q_j \tag{3}$$

Where *K. E.* is the kinetic energy, *P. E.* is the potential energy, $x_j$ is the generalized coordinate, $Q_j$ is the generalized force associated with generalized coordinate, and R is any dissipative energy. Considering the displacement along the inclined surface x as the only generalized coordinate, then:

1) Identify generalized axis $x$
2) Identify generalized forces associated with generalized axes to be zero if we start from the equilibrium position
3) Kinetic Energy of the system

$$K.E. = \frac{1}{2}m\ (\dot{x}^2) \tag{4}$$

4) Potential Energy

$$P.E. = \frac{1}{2}k(x^2) \tag{5}$$

5) Resistance R as dissipative energy

$$R = \frac{1}{2}c\ (\dot{x}^2) \tag{6}$$

Substitute from equations (3-5) into equation (6) yields:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{7}$$

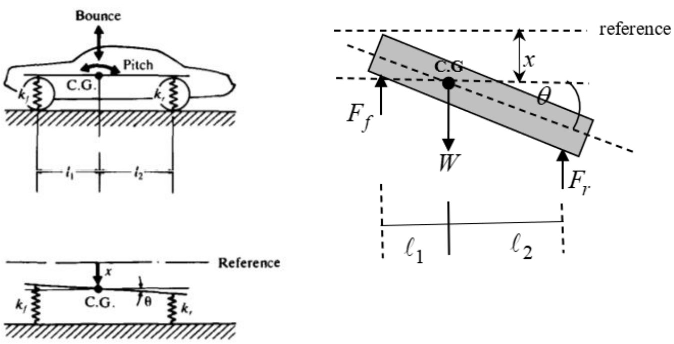
It should be noted that as long as the stiffness elements are linear, we can ignore the gravitational force mg if we measure all displacements from the static-equilibrium positions corresponding to no inputs except gravity [3].

## 2.2 Example 2

A side view of a car of mass on a straight road is shown in Figure 2 (a) [4]. The vertical bounce and rotation of the car body about its center of gravity will be studied. The forward motion of the car which depends mainly on the engine force will not be studied and assumed to be independent of the previous motions. The car body will be assumed rigid during oscillations and the front and rear suspensions are assumed to be linear springs. In addition, the road will be assumed smooth.

### 2.2.1 Newton's method

Figure 1 (b) shows the free body diagram of the car at general vertical and angular displacements. From the geometry the deflection in the forward and rear springs are $(x - l_1\theta)$ and $(x + l_2\theta)$ respectively as $\theta$ is a small angle.




(**a**) (**b**)

Fig. 2 (a) bouncing car on a road [4] (b) free body diagram in case of rotation and vertical bouncing

Following the same steps of the first example and understanding the basic assumptions, such as small angular displacement. The acceleration equation in the vertical direction is

$$m\ddot{x} = W - k_f(x - l_1\theta) - k_r(l_2\theta + x) \tag{8}$$

Where $k_f$ and $k_r$ are the front and rear spring stiffnesses respectively. Similarly, the moment equation about the center of gravity gives the equation of the pitching motion.

$$I_C\ddot{\theta} = k_f(x - l_1\theta)l_1 - k_r(x + l_2\theta)l_2 \tag{9}$$

Where $I_C$ is the mass moment of inertia of the car. As the above equations satisfy the equilibrium conditions, the weight can be eliminated from equation (8) when considering only deviations from equilibrium i.e. $x$ and $\theta$. In such a case the above equations can be rewritten as:

$$m\ddot{x} = -k_f(x - l_1\theta) - k_r(x + l_2\theta) \tag{10}$$

$$I_C\ddot{\theta} = k_f l_1(x - l_1\theta) - k_r l_2(x + l_2\theta) \tag{11}$$

### 2.2.2 Lagrange's Equation

If we assume that the motion starts from the equilibrium position, the gravity effect can be ignored as it will be balanced by the static deflections of the two springs. The same definition of the displacements at each end of the car is applied here. To calculate the equations of motion for the half care suspension system we need to make five steps as follows:

1) Identify Generalized axes *θ, x*

2) Identify generalized forces associated with generalized axes 0, 0,

3) Kinetic Energy of the system

$$K.E. = \frac{1}{2} m\ (\dot{x}^2) + \frac{1}{2} I_C (\dot{\theta}^2) \tag{12}$$

4) Potential Energy of the system

$$P.E. = \frac{1}{2} k_r (x + \ell_2 \theta)^2 + \frac{1}{2} k_f (x - \ell_2 \theta)^2 \tag{13}$$

5) Resistance R (No viscous damping is used)

$$R = 0 \tag{14}$$

The Lagrange's equation is given by [2]:

$$\frac{d}{dt}\left(\frac{\partial K.E.}{\partial \dot{q}_j}\right) - \left(\frac{\partial K.E.}{\partial q_j}\right) + \left(\frac{\partial P.E.}{\partial q_j}\right) + \left(\frac{\partial R}{\partial \dot{q}_j}\right) = Q_j \tag{15}$$

For $\theta$,

$$\frac{d}{dt}\left(\frac{\partial K.E.}{\partial \dot{\theta}_j}\right) - \left(\frac{\partial K.E.}{\partial \theta_j}\right) + \left(\frac{\partial P.E.}{\partial \theta_j}\right) + \left(\frac{\partial R}{\partial \dot{\theta}_j}\right) = Q_j \tag{16}$$

Substituting for kinetic and potential energies into Lagrange's equation and after some algebraic manipulation yields:

$$I_C \ddot{\theta} + k_r \ell_2 (x + \ell_2 \theta) - k_f \ell_1 (x - \ell_1 \theta) = 0 \tag{17}$$

For *x*:

$$m\ \ddot{x} + k_r (x + \ell_2 \theta) + k_f (x - \ell_1 \theta) = 0 \tag{18}$$

If the motion starts from reference position, the right-hand side of equation (18) will be *W* instead of zero.

## 2.3 Example 3

The 4 degrees of freedom idealized model of a half car suspension system is shown in Figure 3. These four degrees of freedom are vertical translation and plane rotation of the car body and vertical translation of each axle [5], where: $m_1$, $m_2$, $m_3$ is car body, front axle, rear axle masses, respectively, Ic is car body moment of inertia for plane rotation about center of mass, $x_1$ and θ are car body vertical translation of center of mass $x_1$, and rotation θ which is assumed small, $x_2$ and $x_3$ are vertical translations of front and rear axles. k's, and c's are main suspension and tire spring and damping constants, $\alpha_2$ and $\alpha_3$ are distances from car body center of mass to point of attachment of front suspension and rear suspension respectively, $x_f(t)$ is instantaneous height of road relative to a smooth road under front wheel and $x_r(t)$ is instantaneous height of road relative to a smooth road under rear wheel.

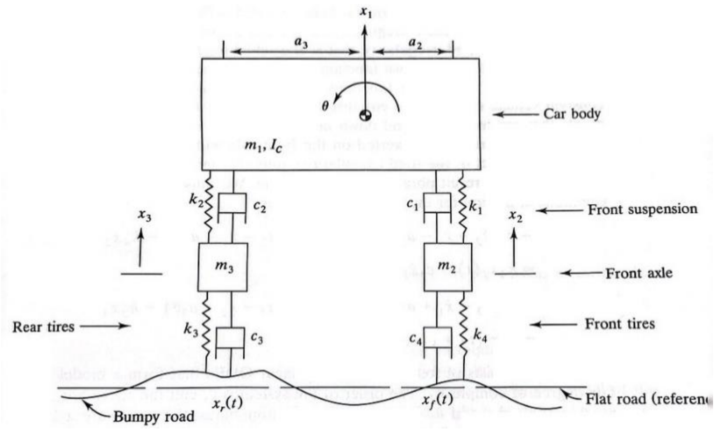



Fig. 3 Idealized model of a 4 degrees of freedom automobile-suspension system [5]

This example is more general than the previous example, as it includes the masses of axles, spring like behavior of tires, and all damping effects. Thus, it approaches more the real half car suspension system and let students link their study in engineering mechanics to real life problems [6]. Also, in this example, it becomes more difficult for a student to imagine the motion of each individual mass apart from other masses.

2.3.1 Newton's method

Since the system is of four degrees of freedom a student needs to write four acceleration equations. After sketching the free body diagram of the car body and each axle mass, we can apply Newton's second law to obtain the equations of motion. In this example, we will start directly assuming all displacements are taken from equilibrium condition. As a result, the gravitational force of any mass will disappear from its motion equations. Solution steps can be listed as follows:

1) Indicated the positive displacement direction of each mass, either translational or rotation
2) Draw the free body diagram for each mass.
3) Determine deflection in each spring and displacement derivative in each damper based on step 1.
4) Start writing the equation of motion of each mass according to the order of variable in the state vector.

Applying Newton's second law on the car body in the vertical direction gives

$$m_1\ddot{x}_1 = c_1(\dot{x}_2 - \dot{x}_1 - a_2\dot{\theta}) + c_2(\dot{x}_3 - \dot{x}_1 + a_3\dot{\theta}) + k_1(x_2 - x_1 - a_2\theta) + k_2(x_3 - x_1 + a_3\theta) \quad (19)$$

considering clockwise direction as positive, the moment equation about the body center leads

$$I_c\ddot{\theta} = c_1a_2(\dot{x}_2 - \dot{x}_1 - a_2\dot{\theta}) - c_2a_3(\dot{x}_3 - \dot{x}_1 + a_3\dot{\theta}) + k_1a_2(x_2 - x_1 - a_2\theta) - k_2a_3(x_3 - x_1 + a_3\theta) \quad (20)$$

Equations of motions of the masses $m_2$, $m_3$ will be

$$m_2\ddot{x}_2 = -c_1(\dot{x}_2 - \dot{x}_1 - a_2\dot{\theta}) + c_4(\dot{x}_f - \dot{x}_2) - k_1(x_2 - x_1 - a_2\theta) + k_4(x_f(t) - x_2) \quad (21)$$

$$m_3\ddot{x}_3 = -c_2(\dot{x}_3 - \dot{x}_1 + a_3\dot{\theta}) + c_3(\dot{x}_r - \dot{x}_3) - k_2(x_3 - x_1 + a_3\theta) + k_3(x_r(t) - x_3) \quad (22)$$

2.3.2 Lagrange's method

To calculate the equations of motion for the half car suspension system we need to make five steps as follows:

1) Identify generalized axes $x_1, \theta, x_2, x_3$
2) Identify generalized forces associated with generalized axes to be equal to zero.
3) Kinetic Energy of the system

$$K.E. = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} I_C \dot{\theta}^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2 \tag{23}$$

4) Potential Energy of the system

$$P.E. = \frac{1}{2} k_1 (x_1 + a_2\theta - x_2)^2 + \frac{1}{2} k_4 (x_2 - x_f(t))^2 + \frac{1}{2} k_2 (x_1 + a_3\theta - x_3)^2 + \frac{1}{2} k_3 (x_3 - x_r(t))^2 \tag{24}$$

5) Resistance R

$$R = \frac{1}{2} c_1 (\dot{x}_1 + a_2\dot{\theta} - \dot{x}_2)^2 + \frac{1}{2} c_4 (\dot{x}_2 - \dot{x}_f(t))^2 + \frac{1}{2} c_2 (\dot{x}_1 - a_3\dot{\theta} - \dot{x}_3)^2 + \frac{1}{2} c_3 (\dot{x}_3 - \dot{x}_r(t))^2 \tag{25}$$

Lagrange's equation for $\theta$,

$$\frac{d}{dt}\left(\frac{\partial K.E.}{\partial \dot{\theta}_j}\right) - \left(\frac{\partial K.E.}{\partial \theta_j}\right) + \left(\frac{\partial P.E.}{\partial \theta_j}\right) + \left(\frac{\partial R}{\partial \dot{\theta}_j}\right) = Q_j \tag{26}$$

Upon substituting for Kinetic and Potential energies into Lagrange's equation and after some algebraic manipulation yields:

$$I_C \ddot{\theta} + c_1 a_2 (\dot{x}_1 + a_2\dot{\theta} - \dot{x}_2) - c_2 a_3 (\dot{x}_1 - a_3\dot{\theta} - \dot{x}_3) + k_1 a_2 (x_1 + a_2\theta - x_2) - k_2 a_3 (x_1 - a_3\theta - x_3) = 0 \tag{27}$$

For $x_1$:

$$m_1 \ddot{x}_1 + c_1 (\dot{x}_1 + a_2\dot{\theta} - \dot{x}_2) + c_2 (\dot{x}_1 - a_3\dot{\theta} - \dot{x}_3) + k_1 (x_1 + a_2\theta - x_2) + k_2 (x_1 - a_3\theta - x_3) = 0 \tag{28}$$

For $x_2$:

$$m_2 \ddot{x}_2 + c_1 (\dot{x}_2 - \dot{x}_1 - a_2\dot{\theta}) + c_4 \dot{x}_2 + k_1 (x_2 - x_1 - a_2\theta) + k_4 x_2 = k_4 x_f(t) + c_4 \dot{x}_f(t) \tag{29}$$

For $x_3$:

$$m_3 \ddot{x}_3 + c_2 (\dot{x}_3 - \dot{x}_1 + a_3\dot{\theta}) + c_3 \dot{x}_3 + k_2 (x_3 - x_1 + a_3\theta) + k_3 x_3 = k_3 x_r(t) + c_3 \dot{x}_r(t) \tag{30}$$

## 2.4 Example 4 Electrical System

The standard way of modelling electrical circuits is applying Kirchhoff's law for any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node; or equivalently: *The algebraic sum of currents in a network of conductors meeting at a point is zero*. Thus, we assume the current passes through each portion of the electric circuit and write down the components equations (Capacitance, Inductance and Resistance), then applying Kirchhoff's law to find the governing equation of the system. For the comparison, the governing equation of the electric circuit will be obtained using Kirchhoff's law and Lagrange's equation as well.

Wells in 1938 briefly presented a convenient form of the Lagrangian equation applicable for Electrical circuits and illustrated their use with a number of examples [7]. Panuluh and Damanik formulated the Lagrangian for the LC, RC, RL, and RLC circuits by using the analogy concept with the mechanical problem in classical mechanics formulations [8]. They showed that the Lagrangian for LC and RLC circuits are composed of terms that can be assigned as kinetic energy and potential energy terms in corresponding with the Lagrangian of a physical system in classical mechanics. Sira-Ramirez et al., 1996 used Lagrangian formalism to model DC-to-DC power converters with switch-regulation [9].

2.4.1 Kirchhoff's Law

Consider the following electric circuit which consists of two resistances $R$ and $R_L$, Capacitance $C$ and Inductance L with external voltage source $U_a(t)$. It is required to find the governing equation of the electric circuit in state-space format:

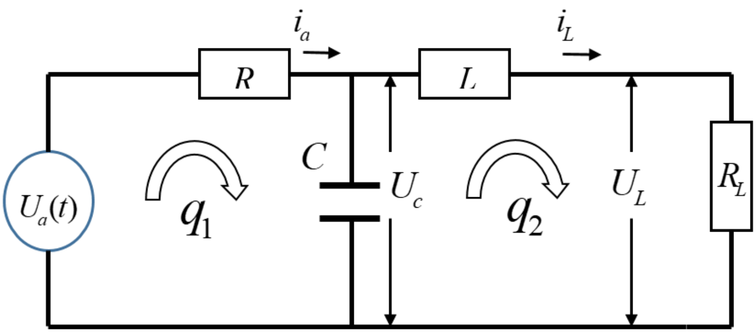



Fig. 4 Electric circuit model [9,10]

Component's Equation

$$i_R = \frac{U_a - U_c}{R}, \quad i_c = CDU_c \text{ with } \mathrm{U_c}(0), \quad i_L = \frac{1}{LD}(U_c - U_L) \text{ with } \mathrm{i_L}(0), \quad i_{RL} = \frac{U_L}{R_L} \tag{31}$$

Node Equations

$$i_R = I_c + i_L, \qquad i_{RL} = i_L \tag{32}$$

If we select $U_c$ and $i_L$ as two state variables and by substituting from the component's equation into the first node equations yields:

$$\frac{U_a - U_c}{R} = CDU_c + i_L \tag{33}$$

Then after some algebraic manipulation:

$$DU_c = \frac{1}{C}\left(\frac{U_a}{R} - \frac{U_c}{R} - i_L\right) \tag{34}$$

Substitute from the component's equation into the second node equation of equation (31) yields:

$$Di_L = \frac{1}{L}(U_c - i_L R_L) \tag{35}$$

Equations (34) and (35) are the state-space form of the electric circuit. In matrix format:

$$\begin{bmatrix} DU_c \\ Di_L \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R_L}{L} \end{bmatrix} \times \begin{bmatrix} U_c \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{U_a}{CR} \\ 0 \end{bmatrix} \tag{36}$$

2.4.2 Lagrange's Equation

The application of Lagrange's equation can be extended to the modelling of electrical circuits using the same principle of energy and system similarity. For electrical systems, electrical charges may also serve as appropriate generalized coordinate. The word generalized coordinates enable us to select suitable parameters that are convenient to describe the dynamics of the system. Consider the following similarity quantities:

$$K.\,E. = \frac{1}{2}L\dot{q}^2,\ P.\,E. = \frac{1}{2}Lq^2,\ R = \frac{1}{2}R\dot{q}^2,\ i = \dot{q} \text{ and } v_c = \frac{q}{C} \tag{37}$$

Where L is the inductance of the circuit, C is the capacitance and R is the resistance. If we consider the generalized force to be the external input voltage to the circuit $Q=U_a(t)$, then Lagrange's equation for electrical systems becomes:

$$\frac{d}{dt}\left(\frac{\partial K.E.}{\partial \dot{q}_j}\right) - \left(\frac{\partial K.E.}{\partial q_j}\right) + \left(\frac{\partial P.E.}{\partial q_j}\right) + \left(\frac{\partial R}{\partial \dot{q}_j}\right) = u(t) \tag{38}$$

Choosing $q_1$ and $q_2$ as the generalized independent coordinates, then:

$$i_a = \dot{q}_1,\quad i_L = \dot{q}_2,\quad U_a(t) = Q_1,\quad Q_2 = 0 \tag{39}$$

The total magnetic energy (Kinetic energy) of the circuit is $K.E. = \frac{1}{2}L\dot{q}^2$, The total electrical energy (Potential energy) of the circuit is $P.E. = \frac{1}{2}\frac{(q_1 - q_2)^2}{C}$, The total dissipative energy of the circuit is $R = \frac{1}{2}R\dot{q}_1^{\,2} + \frac{1}{2}R_L\dot{q}_2^{\,2}$

Substituting the three types of energy into Lagrange's Equation (16) for $q_1$ and $q_2$ yields the two equations of motion as follows:

$$R\dot{q}_1 + \frac{q_1 - q_2}{C} = U_a(t) \tag{40}$$

$$L\ddot{q}_2 + R_L\dot{q}_2 + \frac{(q_2 - q_1)}{C} = 0 \tag{41}$$

By using Kirchhoff's law, one can get the same equations of motion (33 and 34) [10].

$$Ri_a + U_c = U_a(t),\ R(i_c + i_L) + U_c = U_a(t),$$
$$(i_c + i_L) = \frac{1}{R}(U_a(t) - U_c)$$
$$CDU_c = \frac{1}{R}(U_a(t) - U_c) - i_L,$$

Then $DU_c = \frac{1}{C}(\frac{U_a(t)}{R} - \frac{U_c}{R} - i_L)$

For the second equation, $LDi_L + R_L i_L + U_c = 0,$

then $Di_L = \frac{1}{L}(U_c - R_L i_L)$

It should be noticed from the first three previous examples that the application of Newton's second law and Lagrange's equation give the same equations of motion exactly. For the single degree of freedom system, Newton's method is easier and straight forward with less effort compared to Lagrange's equation. When the number of degrees of freedom increases, Newton's second law of motion becomes more difficult and needs more kinematic analysis before applying the three equations for X, Y and moment to find the equations of motion. Lagrange's equation for multiple degrees of freedom is more applicable beacuase the concept of generalized coordinates makes it suitable for obtaining the equations of motion even for electrical systems as well using system similarity. With basic differentiation and some algebraic manipulations, the students can find the equations of motion directly.

Applying Newton's second law of motion enables the students to understand the physics of the problem and how the internal forces that should be assumed in opposite directions can affect the equations of motion. It should be mentioned that for systems with springs, students can easily assign the spring force opposite to the possible direction of motion when they apply Newton's second law of motion. However, for multiple degrees of freedom systems, the students feel some difficulties in assuming the correct direction for the spring and damper forces. The confusion is always there: Is it ( $x_2 - x_1$) or ( $x_1 - x_2$). The correct approach is to assume that either $x_2 \succ x_1$ and start from $x_2$ and assign the force in opposite direction of motion or $x_1 \succ x_2$ and start from $x_1$. The beautiful thing in applying Lagrange's equation is we do not have this problem at all. Starting with any of these two assumptions will not change the terms of the equations of motion. This is also the case for dissipative viscous damping as well.

## Conclusions

Deriving equation of motion using momentum principles was found straightforward in simple systems up to two degrees of freedom. For systems with 3 DOF and higher it is recommended to use Lagrange's Equation or other energy techniques. As a vector equation, the momentum-based principle requires coordinates to express components of forces and accelerations. In energy principles, represented here by Lagrange's equation, the most important step is to understand the motion in terms of the generalized coordinates in general. Then, forces and other quantities are needed to be expressed as functions of these coordinates or their derivatives to be included into Lagrange's equation. Governing equations of electric circuits can also be obtained using Lagrange's equation. But still to reach final state-space form, a traditional Kirchhoff's law should be applied.

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