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Research Article

HIGH ORDER CUBIC B-SPLINE GALERKIN METHOD FOR ADVECTION EQUATION

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ABSTRACT. In this study, the Advection equation will be solved numerically using the cubic B-spline Galerkin finite-element method, based on second, fourth and sixth order single step methods for time integration. The test problem is studied, and accuracy of the numerical results are measured by the computing the maximum error norm for the proposed methods. The numerical results of this study demonstrate that the proposed three algorithms especially the sixth order single step method are a remarkably successful numerical technique for solving the Advection equation.

Keywords: *Galerkin method, Advection equation, Cubic B-spline*

INTRODUCTION

The Advection equation is the basis of many physical and chemical phenomena. Various numerical techniques have been developed and compared for solving the one dimensional Advection equation with constant coefficient so far [1,2,3]. This study presents high order time discretization numerical method for the Advection equation. The main idea of using this method is to obtain high-order approximate solution for Advection equation by using Taylor series expansion. The structure of the study is as follows. In the next section, after the time discretization of the Advection equation is performed by using higher accurate finite difference method, a finite element space discretization is used to obtain a system of algebraic equation. In the numerical experiment section, proposed methods are tested for the two test problems and finally, a summary of main findings of the work is presented in the last section.
We consider the following one-dimensional Advection equation

$$\boldsymbol{u_t + \alpha u_x = 0, a \le x \le b} \tag{1}$$

with the boundary conditions

$$\boldsymbol{u(a,t) = u(b,t) = 0, t \in [0,T]} \tag{2}$$

and initial condition

$$\boldsymbol{u(x,0) = f(x), a \le x \le b} \tag{3}$$

in a restricted solution domain over the space/time interval $[\boldsymbol{a},\boldsymbol{b}] \times [\boldsymbol{0},\boldsymbol{T}]$. In the one-dimensional linear Advection equation, $\boldsymbol{\alpha}$ is the steady uniform fluid velocity and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ is a function of two independent variables $\boldsymbol{t}$ and $\boldsymbol{x}$, which generally denote time and space, respectively.

APPLICATION OF THE METHOD

For computational work, the space-time plane is discretized by grids with the time step $\boldsymbol{k}$ and space step $\boldsymbol{h}$. The exact solution of the unknown function at the grid points is denoted by

$$u(x_m, t_n) = u_m^n, m = 0, 1, \ldots, N; \quad n = 0, 1, 2, \ldots$$

where $x_m = a + mh$, $t_n = nk$ and the notation U_m^n is used to represent the numerical value of u_m^n.

Time Discretization

Using the Advection equation of the form

$$u_t = -\alpha u_x \tag{4}$$

and the following one-step method

$$u^{n+1} = u^n + \theta_1 u_t^{n+1} + \theta_2 u_t^n + \theta_3 u_{tt}^{n+1} + \theta_4^n u_{tt}^n + \theta_5 u_{ttt}^{n+1} + \theta_6^n u_{ttt}^n, \tag{5}$$

we have the time discretization of the Eq. (1). Taking

$$\theta_1 = \theta_2 = k/2, \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$$

in (5) , the proposed method is of order 2 (M1) known as Crank-Nicolson method (CN method) and then taking

$$\theta_1 = \theta_2 = k/2, \theta_3 = -k^2/12, \theta_4 = k^2/12, \theta_5 = \theta_6 = 0$$

in (5), the method is of order 4 (M2). Finally taking

$$\theta_1 = \theta_2 = k/2, \theta_3 = -k^2/10,\ \theta_4 = k^2/10, \theta_5 = k^3/120, \theta_6 = k^3/120$$

in (5), the method is of order 6 (M3). Using the Eq. (4) then we have the following relations:

$$u_{tt} = -\alpha(u_t)_x = -\alpha(-\alpha u_x)_x = \alpha^2 u_{xx},$$
$$u_{ttt} = \alpha^2(u_t)_{xx} = \alpha^2(-\alpha u_x)_{xx} = -\alpha^3 u_{xxx}.$$

Using u_t, u_{tt}, u_{ttt} in the proposed time discretization method for the Advection equation, we have

$$\begin{aligned} u^{n+1} + \alpha\theta_1(u_x)^{n+1} - \alpha^2\theta_3(u_{xx})^{n+1} + \alpha^3\theta_5(u_{xxx})^{n+1} \\ = u^n - \alpha\theta_2(u_x)^n + \alpha^2\theta_4(u_{xx})^n - \alpha^3\theta_6(u_{xxx})^n. \end{aligned} \tag{6}$$

The interval $[a, b]$ is divided into uniformly sized N finite sub-elements of equal length h at the knots

$$a = x_0 < x_1 < \cdots < x_N = b.$$

On this partition, the cubic B-splines φ_m, $m = -1, \ldots, N + 1$, have the following form [4,5,6]:

$$\varphi_m(x) = \frac{1}{h^3}\begin{cases} (z_{m-2})^3 & , \ x_{m-2} \le x < x_{m-1} \\ h^3 + 3h^2 z_{m-1} + 3h(z_{m-1})^2 - 3(z_{m-1})^3 & , \ x_{m-1} \le x < x_m \\ h^3 - 3h^2 z_{m+1} + 3h(z_{m+1})^2 + 3(z_{m+1})^3 & , \ x_m \le x < x_{m+1} \\ -(z_{m+2})^3 & , \ x_{m+1} \le x < x_{m+2} \\ 0 & , \ \text{otherwise} \end{cases} \tag{7}$$

where $z_m = (x - x_m)$. The set of cubic B-splines $\varphi_m(x)$, $m = -1, \ldots, N+1$ forms a basis over the space interval $a \le x \le b$. Over the problem domain, the approximate solution $U(x,t)$ to the exact solution $u(x,t)$ can be written as a combination of the cubic B-splines

$$U(x,t) = \sum_{j=-1}^{N+1} \delta_j \varphi_j \tag{8}$$

where δ_j are time dependent unknown parameters which will be determined from the Galerkin method and the boundary and initial conditions. Since the cubic B-spline functions (7) and its first second derivatives are continuous, trial solutions (8) have continuity up to second order. Using (7-8), the nodal values U and its first and second space derivatives at the knots x_m are obtained as

$$\begin{aligned} U_m = U(x_m) &= \delta_{m+1} + 4\delta_m + \delta_{m-1}, \\ U'_m = U'(x_m) &= \frac{3}{h}(\delta_{m+1} - \delta_{m-1}), \\ U''_m = U''(x_m) &= \frac{6}{h^2}(\delta_{m-1} - 6\delta_m + \delta_{m+1}). \end{aligned} \tag{9}$$

A typical finite interval $[x_m, x_{m+1}]$ is mapped to the interval $[0, h]$ by a local coordinate transformation defined by $\xi = x - x_m$. Therefore cubic B-spline shape functions in terms of ξ over the element $[0, h]$ can be given as

$$\begin{aligned} \varphi_{m-1}(\xi) &= \left(1 - \frac{\xi}{h}\right)^3, \\ \varphi_m(\xi) &= 4 - 6\frac{\xi^2}{h^2} + 3\frac{\xi^3}{h^3}, \\ \varphi_{m+1}(\xi) &= 1 + 3\frac{\xi}{h} + 3\frac{\xi^2}{h^2} - 3\frac{\xi^3}{h^3}, \\ \varphi_{m+2}(\xi) &= \frac{\xi^3}{h^3}. \end{aligned} \tag{10}$$

Combination of the element shape functions φ_i together with element time parameters δ_i, $i = m-2, \ldots, m+3$ gives an approximation for the typical element $[0, h]$

$$U^e = U(\xi, t) = \sum_{j=m-1}^{m+2} \delta_j(t) \varphi_j(\xi) \tag{11}$$

Applying Galerkin method to Eq. (5) with weight function $W(x)$ leads to the equation:

$$\begin{aligned} &\int_a^b W(x)(u^{n+1} + \alpha\theta_1(u_x)^{n+1} - \alpha^2\theta_3(u_{xx})^{n+1} + \alpha^3\theta_5(u_{xxx})^{n+1})dx \\ &\quad = \int_a^b W(x)(u^n - \alpha\theta_2(u_x)^n + \alpha^2\theta_4(u_{xx})^n - \alpha^3\theta_6(u_{xxx})^n)dx. \end{aligned} \tag{12}$$

In the above Galerkin method formulation, weight functions $W(x)$ and exact solution are replaced with cubic B-splines shape functions (10) and approximation given by (11), respectively. Integrating by parts and using the boundary conditions yields a fully discrete approximation obtained over the element $[\mathbf{0},\boldsymbol{h}]$ as:

$$\begin{aligned}&\sum_{j=m-1}^{m+2}\left(\int_0^h \varphi_i\left(\varphi_j+\alpha\theta_1\varphi_j'-\alpha^2\theta_3\varphi_j''\right)d\xi-\alpha^3\theta_5\int_0^h \varphi_i'\varphi_j''d\xi\right)\delta_j^{n+1}\\ &-\sum_{j=m-1}^{m+2}\left(\int_0^h \varphi_i\left(\varphi_j-\alpha\theta_2\varphi_j'+\alpha^2\theta_4\varphi_j''\right)d\xi+\alpha^3\theta_6\int_0^h \varphi_i'\varphi_j''d\xi\right)\delta_j^{n}\end{aligned} \quad (13)$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ take only the values $\boldsymbol{m-1,\dots,m+2}$ and $\boldsymbol{m=0,1,\dots,N-1}$ for the typical element $[\mathbf{0},\boldsymbol{h}]$. (13) can be written in matrix form as

$$\begin{aligned}&[A^e+\alpha\theta_1B^e-\alpha^2\theta_3C^e-\alpha^3\theta_5D^e](\delta^e)^{n+1}\\ &\qquad-[A^e-\alpha\theta_2B^e+\alpha^2\theta_4C^e+\alpha^3\theta_6D^e](\delta^e)^{n}\end{aligned} \quad (14)$$

where the dimension of the element matrices $\boldsymbol{A^e}$, $\boldsymbol{B^e}$, $\boldsymbol{C^e}$, $\boldsymbol{D^e}$, $\boldsymbol{E^e}$ are $\mathbf{4}\times\mathbf{4}$, and the element matrices and element parameters are

$$A_{i,j}^e=\int_0^h \varphi_i\varphi_j d\xi,\quad B_{i,j}^e=\int_0^h \varphi_i\varphi_j' d\xi,\quad C_{i,j}^e=\int_0^h \varphi_i\varphi_j'' d\xi,\quad D_{i,j}^e=\int_0^h \varphi_i'\varphi_j'' d\xi,$$

Assembling contributions from all elements, (12) leads to the following linear system for the time evolution of $\boldsymbol{\delta}$:

$$[A+\alpha\theta_1B-\alpha^2\theta_3C-\alpha^3\theta_5D]\delta^{n+1}=[A-\alpha\theta_2B+\alpha^2\theta_4C+\alpha^3\theta_6D]\delta^{n}. \quad (15)$$

The linear system (15) consists of $\boldsymbol{N+3}$ linear equations in $\boldsymbol{N+3}$ unknowns $\left(\delta_{-1}^{n+1},\dots,\delta_{N+1}^{n+1}\right)$. After initial vector $\boldsymbol{\delta^0}=(\delta_{-1}^{0},\dots,\delta_{N+1}^{0})$ is found with the help of the boundary and initial conditions, $\boldsymbol{\delta^{n+1}},(\boldsymbol{n}=\mathbf{0},\mathbf{1},\dots)$ unknown vectors can be found repeatedly by solving the recurrence relation (14) using previous $\boldsymbol{\delta^n}$ unknown vector.

TEST PROBLEM

For the test problem, accuracy of the proposed three algorithms is worked out by measuring error norm $\boldsymbol{L_\infty}$

$$L_\infty=\max_m|u_m-U_m| \quad (16)$$

In this test problem, the Advection equation has the exact solution

$$u(x,t)=10\,exp\left(-\frac{(x-\tilde{x}_0-\alpha t)^2}{2\rho^2}\right).$$

The numerical simulation is accomplished with flow velocity $\boldsymbol{\alpha}=\mathbf{0.5}\boldsymbol{m/s}$, initial peak location $\tilde{\boldsymbol{x}}_{\mathbf{0}}=\mathbf{2}\boldsymbol{km}$ and $\boldsymbol{\rho}=\mathbf{264}$ by the terminating time $\boldsymbol{t}=\mathbf{10000}\boldsymbol{s}$. Therefore the initial condition $\boldsymbol{u(x,0)}$ is propagated in a long channel without change in shape or size by the time $\boldsymbol{t}=\mathbf{10000}\boldsymbol{s}$ with flow velocity $\boldsymbol{\alpha}=\mathbf{0.5}\boldsymbol{m/s}$. So initial condition travels from the initial position to a distance of $\mathbf{5}\boldsymbol{km}$ and the peak value of the solution remain constant $\mathbf{10}$ for all time. After the program run up to time $\boldsymbol{t = 10000s}$, initial solutions and waves at various times are depicted in Fig. 1 for the M3 with $\boldsymbol{h = \Delta t = 10}$. It can be seen from the figure that wave propagates without any change in its shape.

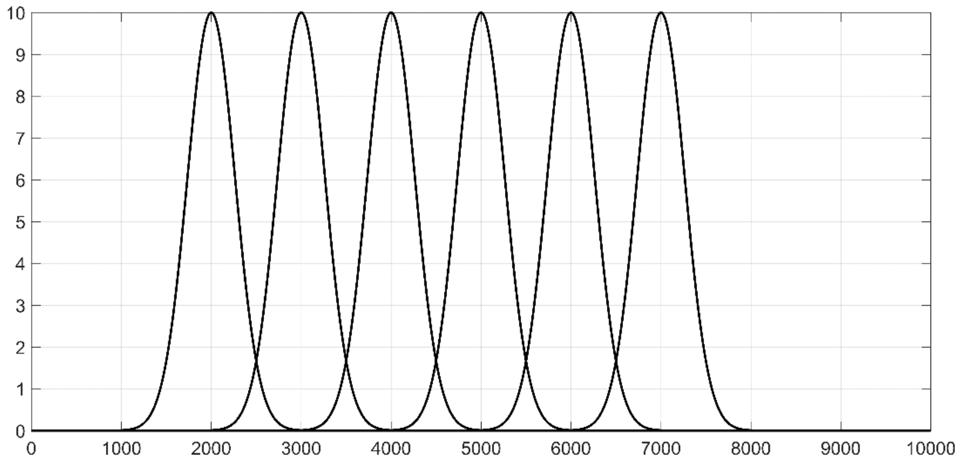

Fig. 1. *Waves at t=0,2000,4000,6000,8000,10000.*

The error norms L_∞ for the proposed three methods are listed in Table 1. According to the table, when time and space steps are reduced from 200 to 10, the error norms decrease for all algorithms. Among the suggested methods, the sixth order method gives the least error as expected.

Table 1. *The error norms for the proposed three methods*

h=k	*M1*	*M2*	*M3*
200	*2.42*	1.14×10^{-1}	8.32×10^{-2}
100	7.62×10^{-1}	1.96×10^{-3}	1.82×10^{-4}
50	1.98×10^{-1}	1.22×10^{-4}	5.50×10^{-7}
20	3.13×10^{-2}	3.13×10^{-6}	5.02×10^{-10}
10	7.82×10^{-3}	1.96×10^{-7}	5.40×10^{-12}

CONCLUSION

The high-order Galerkin finite-element method based on Taylor series expansion for the time discretization and cubic B-spline functions for the space discretization was proposed to solve numerically the Advection equation. The test problem was simulated well with the proposed three algorithms. As expected, it was seen from the test problem that the third algorithm with the highest accuracy gave better results. Consequently, the numerical result of this study demonstrates that the proposed algorithms especially the sixth order method are a remarkably successful numerical technique for solving the Advection equation. It can also be efficiently applied to similar physically important equations.

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