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Application of Homotopy Perturbation Method on Some Linear and Nonlinear Periodic Equations

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Abstract

In this paper, the homotopy perturbation method (HPM) which doesn't small parameter is applied to solve the linear and nonlinear parabolic equations. The HPM deforms a difficult problem into a simple problem which can be easily solved. It is implemented with appropriate initial conditions. Comparison of the applied methods with exact solutions reveals that the method is tremendously effective.

Keywords: Homotopy perturbation method, parabolic equations, periodic equation, linear and nonlinear

INTRODUCTION

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [1-3]. Unlike classical techniques, the homotopy perturbation method leads to an analytical approximate and exact solutions of the nonlinear equations easily and elegantly without transforming the equation or linearizing the problem and with high accuracy, minimal calculation and avoidance of physically unrealistic assumptions. As a numerical tool, the method provide us with numerical solution without discretization of the given equation, and therefore, it is not effected by computation round-off errors and one is not faced with necessity of large computer memory and time. This technique has been employed to solve a large variety of linear and nonlinear problems [4-10].

In the present study, homotopy perturbation method has been applied to solve the parabolic equations. The numerical results are compared with the exact solutions. It is shown that the errors are very small. However, recently, Adomian decomposition method has was applied for approximating the solution of the parabolic equations [11].

BASIC IDEA OF HE'S HOMOTOPY PERTURBATION METHOD

To illustrate the basic ideas of this method, we consider the following nonlinear differential Equation:

$$A(u) - f(r) = 0, r \in \Omega \tag{1}$$

Considering the boundary conditions of:

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma \tag{2}$$

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be, generally divided into two parts of L and N, where L is the linear part, while N is the nonlinear one. Eq. (7) can, therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

By the homotopy technique, we construct a homotopy as $v(r,p): \Omega \times [0,1] \rightarrow R$ which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0,\ p\in[0,1],\ r\in\Omega \tag{4}$$

Or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0 \tag{5}$$

Where $p\in[0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (2) which satisfy the boundary conditions. Obviously, considering Eqs. (10) and (11), we will have:

$$H(v,0)=L(v)-L(u_0)=0 \tag{6}$$

$$H(v,1)=A(v)-f(r)=0 \tag{7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopy.

According to HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of Eqs. (10) and (11) can be written as a power series in p:

$$v=v_0+pv_1+p^2v_2+\cdots \tag{8}$$

Setting $p=1$ results in the approximate solution of Eq.(7):

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+.... \tag{9}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which lessens the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques.

The series (15) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$. The following opinions are suggested by He:

1) The second derivative of $N(v)$ with respect to v must be small because the parameter p may be relatively large, i.e. $p\to 1$.

2) The norm of $L^{-1}\partial N/\partial v$ must be smaller than one so that the series converges.

APPLICATIONS OF HPM

In this section, we demonstrate the main algorithm of homotopy perturbation method on linear and nonlinear parabolic equations with initial condition, namely we consider:

$$\frac{du}{dt}=\frac{d^2u}{dx^2}+\Phi(u)+g(x,t),(x,t)\in[a,b]x(0,T) \tag{10}$$

with the initial condition

$$u(x,0)=f(x) \tag{11}$$

where φ is a function of u. We are looking for the solution satisfying Eqs.(1) -(2).

Example1.

This problem was used by Hopkins and Wait [12] to provide an example of a problem with a nonlinear source term:

$$\frac{du}{dt}=\frac{d^2u}{dx^2}+e^{-u}+e^{-2u},(x,t)\in[a,b]x(0,T) \tag{12}$$

with the initial condition $u(x,0)=Ln(x+2)$. In this example we have $\Phi(u)=e^{-u}+e^{-2u}$, $g(x,t)=0, f(x)=Ln(x+2)$

We construct the following homotopy:

$$\frac{du}{dt}-\frac{du_0}{dt}=p(\frac{d^2u}{dx^2}+e^{-u}+e^{-2u}-\frac{du_0}{dt}) \tag{13}$$

Assume the solution of Eq. (13)to be in the form:

$$u=u_0+pu_1+p^2u_2+p^3u_3+\dots \tag{14}$$

Substituting (14) into (13) and equating the coefficients of like powers p, we get the following set of differential equations:

$$\begin{aligned}
&p^0:\frac{du_0}{dt}-\frac{du_0}{dt}=0\\
&p^1:\frac{du_1}{dt}=\frac{d^2u_0}{dx^2}+e^{-u_0}+e^{-2u_0}-\frac{du_0}{dt}\\
&p^2:\frac{du_2}{dt}=\frac{d^2u_1}{dx^2}+u_1(-e^{-u_0}-2e^{-2u_0})\\
&p^3:\frac{du_3}{dt}=\frac{d^2u_2}{dx^2}+(-u_2+\frac{1}{2}v_1^2)e^{-u_0}+(-2u_2+2u_1^2-\frac{1}{48}u_2u_1^2)e^{-2u_0}
\end{aligned} \tag{15}$$

Solving the above equations , we obtain

$$\begin{aligned}
&u_0=Ln(x+2),\\
&u_1=\frac{t}{x+2},\\
&u_2=\frac{-t^2}{2(x+2)^2},\\
&u_3=\frac{t^3}{3(x+2)^3}\\
&u_n=\frac{(-1)^{n+1}t^n}{n(x+2)^n},
\end{aligned} \tag{16}$$

Therefore from the results we can obtain

$$\begin{aligned}
u(x,t)&=Ln(x+2)+\frac{t}{x+2}-\frac{t^2}{2(x+2)^2}+\frac{t^3}{3(x+2)^3}+\dots+\frac{(-1)^{n+1}t^n}{n(x+2)^n}+\dots\\
&=Ln(x+2)+Ln(\frac{t}{x+2}+1)=Ln(x+t+2)
\end{aligned} \tag{17}$$

which is the exact solution of the problem. The absolute error for various values of x, t and M (number of terms) are also tabulated in Table 1.

Example 2.

This problem was used by Lawson and et. al. [12] as the form:

$$\frac{du}{dx}=\frac{d^2u}{dx^2}+(\pi^2-1-p)u+(pe^{-t}+e^{-pt}),(x,t)\in[a,b]x(0,T) \quad (18)$$

with the initial condition

$$u(x,0)=2\sin(\pi x) \quad (19)$$

In this example example we have $\Phi(u)=(\pi^2-1-p)u$, $g(x,t)=pe^{-t}+e^{-pt}, f(x)=2\sin(\pi x)$.

We construct the following homotopy:

$$\frac{du}{dt}-\frac{du_0}{dt}=p(\frac{d^2u}{dx^2}+(\pi^2-1-p)u+(pe^{-u}+e^{-pt})-\frac{du_0}{dt}) \quad (20)$$

Substituting (5) into (20) and equating the coefficients of like powers *p*, we get following set of differential equations:

$$p^0:\frac{du_0}{dt}-\frac{du_0}{dt}=0$$

$$p^1:\frac{du_1}{dt}=\frac{d^2u_0}{dx^2}+(\pi^2-1-p)u_0+(pe^{-u}+e^{-pt})-\frac{du_0}{dt}$$

$$p^2:\frac{du_2}{dt}=\frac{d^2u_1}{dx^2}+(\pi^2-1-p)u_1 \quad (21)$$

$$p^3:\frac{du_3}{dt}=\frac{d^2u_2}{dx^2}+(\pi^2-1-p)u_2$$

Solving the above equations, we obtain

$$u_0=[2-pe^{-t}+\frac{1}{p^2}e^{-pt}+(p+\frac{1}{p})t-(p+\frac{1}{p^2})]\sin(\pi x), \quad (22)$$

$$u_1=-(1+p)[2T+pe^{-t}+\frac{1}{p^2}e^{-pt}+(p+\frac{1}{p})t-(p+\frac{1}{p^2})]\sin(\pi x), \quad (23)$$

$$u_2=(1+p)^2[t^2-pe^{-t}-\frac{1}{p^3}e^{-pt}+(p+\frac{1}{p})\frac{t^2}{2!}-(p+\frac{1}{p^2})t+(p+\frac{1}{p^3})]\sin(\pi x), \quad (24)$$

and so on. Therefore from the equations, we have

$$\begin{aligned}u(x,t)&=[2-pe^{-t}+\frac{1}{p^2}e^{-pt}+(p+\frac{1}{p})t-(p+\frac{1}{p^2})]\sin(\pi x)\\&-(1+p)[2T+pe^{-t}+\frac{1}{p^2}e^{-pt}+(p+\frac{1}{p})t-(p+\frac{1}{p^2})]\sin(\pi x)+\\&(1+p)^2[t^2-pe^{-t}-\frac{1}{p^3}e^{-pt}+(p+\frac{1}{p})\frac{t^2}{2!}-(p+\frac{1}{p^2})t+(p+\frac{1}{p^3})]\sin(\pi x)+\dots\\&=(e^{-t}+e^{-pt})\sin\pi x\end{aligned} \quad (25)$$

The absolute error for various values of x, t and M (number of terms) are also tabulated in Table 2.

Table 1. Absolute error for various values of x, t and M (number of terms) for test problem 1.

x/t	0,2	0,4	0,6	0,8	1
M=5					
0,2	8,7288E-8	5,2113E-6	5,5632E-5	2,9414E-4	0,0011
0,4	5,2100E-8	3,1268E-6	3,3532E-5	1,7801E-4	6,4360E-4
0,6	3,2396E-8	1,9530E-6	2,1027E-5	1,1202E-4	4,0630E-4
0,8	2,0859E-8	1,2624E-6	1,3640E-5	7,2890E-5	2,6512E-4
1	1,3842E-8	8,4059E-7	9,1099E-6	4.8819E-5	1,7801E-4
M=10					
0,2	2,9388E-13	5,5942E-10	4,5169E-8	1,0028E-6	1,0989E-5
0,4	1,1369E-13	2,1740E-10	1,7638E-8	3,9323E-7	4,3255E-6
0,6	4,6851E-14	9,1057E-11	7,4176E-9	1,6601E-7	1,8321E-6
0,8	2,1094E-14	4,0656E11	3,3245E-9	7,4639E-8	8,2616E-7
1	9,9920E-15	1,9182E-11	1,5736E-9	3,5433E-8	3,9323E-7
M=20					
0,2	2,2204E-16	3,3307E-16	5,2625E-14	2,1011E-11	2,1403E-9
0,4	1,1102E-16	6,6613E-16	9,1038E-15	3,4535E-12	3,5318E-10
0,6	4,4409E-16	2,2204E-16	1,5543E-15	6,5525E-13	6,7235E-11
0,8	2,2204E-16	2,2204E-16	2,2204E-16	1,4078E-13	1,4458E-11
1	2,2204E-16	6,6613E-16	6,6613E-16	3,4195E-14	3,4537E-12

Table 2. Absolute error for various values of x , t ,p and M (number of terms) for test problem 2

x/t	0,2	0,4	0,6	0,8	1
M=20,p=1					
0,2	4,9449E-13	3,5554E-12	5,4122E-10	2,3279E-11	2,4691E-10
0,4	8,0003E-13	5,7551E-12	8,7571E-10	3,7667E-11	3,9951E-10
0,6	8,0025E-13	5,7527E-12	8,7571E-10	3,7667E-11	3,9951E-10
0,8	4,9438E-13	3,5553E-12	5,4122E-10	2,3280E-11	2,4691E-10
1	1,0304E-28	7,4077E-28	1,1276E-25	4,8503E-27	1,1444E-26
M=10,p=2					
0,2	1,0967E-10	2,0841E-7	1,7609E-5	4,0750E-4	0,0046
0,4	1,7745E-10	3,3721E-7	2,8462E-5	6,5934E-4	0,0075
0,6	1,7745E-10	3,3721E-7	2,8492E-5	6,5934E-4	0,0075
0,8	1,0967E-10	2,0841E-7	1,7609E-5	4,0750E-4	0,0046
1	2,2850E-26	4,3421E-23	3,6689E-21	8,8901E-21	9,6653E-19
M=20,p=3					
0,2	3,8205E-4	5,1061E-4	6,6805E-5	2,8638E-4	3,1596E-4
0,4	6,1818E-4	8,2618E-4	1,0809E-4	4,6338E-4	5,1124E-4
0,6	6,1817E-4	8,2618E-4	1,0809E-4	4,6338E-4	5,1124E-4
0,8	3,8205E-4	5,1061E-4	6,6805E-5	2,8638E-4	3,1596E-4
1	7,9600E-20	1,0639E-19	1,3919E-20	5,9668E-20	6,5830E-20

CONCLUSION

In the present study the homotopy perturbation method was applied on some periodic equations. The solution has been compared with the exact solution. The results show that while the traditional perturbation method depends on small parameter assumption, and the obtained results, in most cases, end up with a non physical result, the numerical method leads to inaccurate results when the equation is intensively dependent on time, while He's homotopy perturbation method (HPM) overcomes completely the above shortcomings, revealing that the HPM is very convenient and effective.

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