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The Orlicz Space of Entire Sequences associated with Multiplier Sequences

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Abstract

Let $\lambda = \{\lambda_1, \lambda_2, \dots\}$ be a fixed sequence of non-zero complex numbers. Γ_M is the vector space of Orlicz space of entire sequences . Let $\Gamma_M(\lambda)$ be the subset of Γ_M for which $\{\lambda_k x_k\} \in \Gamma_M$. In this paper, we are concerned with some properties of $\Gamma_M(\lambda)$. In fact, for $\Gamma_M(\lambda)$ to be equal to Γ_M and for $\Gamma_M(\lambda)$ to be included in $\Gamma_M(\mu)$, the necessary and sufficient conditions are obtained . It is shown that $\Gamma_M(\lambda)$ is a complete metric space if and only if $\lim_{k\to\infty}\inf|\lambda_k|^{1/k} > 0$. Furthermore, conjugate space of $\Gamma_M(\lambda)$ is obtained .

Keywords and Phrases : entire sequences, analytic sequences, Orlicz functions.
AMS Subject Classification : 40A05 , 40C05 , 40D05 , 46E45 .

INTRODUCTION

A complex sequence, whose k^{th} term is x_k is denoted by $\{x_k\}$ or simply by x .

Let w be the set of all sequences $x = \{x_k\}$ of all complex or real numbers and Φ be the set of all finite sequences.

A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\wedge$. A sequence x is called entire sequence if $\lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ .

Orlicz [14] used the idea of Orlicz function to construct the space L^M . Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \le p < \infty)$ Subsequently different classes of Orlicz type sequence spaces have been studied by Parashar and Choudhary [15], Mursaleen *et al* [11], Bektas and Altin [1], Tripathy *et al.* [17]. Rao and Subramanian [2] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref [8].

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0,$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If the convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function, defined Nakano [13] and further discussed by Ruckle [16] , Maddox [12] and many others.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \le p < \infty,$ the spaces ℓ_M coincide with the classical sequence space ℓ_p .

Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$ $\delta^{(n)} = (0,0,\dots,1,0,0,\dots)$, 1 in the n^{th} place and zero's elsewhere .

Definition 1.1: The space consisting of all those sequences x in w such that

$M\left(\frac{|x_k|^{1/k}}{\rho}\right) \to 0,$ as $k \to \infty$ for some arbitrary fixed $\rho > 0$ is denoted by Γ_M , M being an Orlicz function.

In other words $\left\{ M\left(\frac{|x_k|^{1/k}}{\rho}\right) \right\}$ is a null sequence . Γ_M is called the Orlicz space of entire sequences.

Definition 1.2. The space consisting of all those sequences x in w such that

$\sup_k M\left(\frac{|x_k|^{1/k}}{\rho}\right) < \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by $\wedge_M$, M being a Orlicz function. In other words $\left\{M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right\}$ is a bounded sequence. $\wedge_M$ is called the Orlicz space of analytic sequences.

The spaces Γ_M and $\wedge_M$ are the metric spaces with the metric

$$d(x,y) = \sup_k M\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right)$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ_M and $\wedge_M$. Let $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ be a given sequence of complex numbers such that $\lambda_k \neq 0$ for all $k \in \mathbb{N}$. The space $\Gamma_M(\lambda)$ is a metric space with the metric

$$d(x,y) = \sup_k M\left(\frac{|\lambda_k|^{1/k} |x_k - y_k|^{1/k}}{\rho}\right), \text{ for all } x = \{x_k\} \text{ and}$$

$y = \{y_k\}$ in $\Gamma_M(\lambda)$.

MAIN RESULTS

Proposition 1: $\Gamma_M(\lambda) = \Gamma_M$ f and only f $\lambda \in \wedge$, where $\wedge$ is vector space of all analytic sequences.

Proof : Suppose that $\lambda \in \wedge$. Always $\Gamma_M(\lambda) \subset \Gamma_M$ (1.1)

Since $\lambda \in \wedge$, we have $\lambda x \in \Gamma_M$, for every $x \in \Gamma_M$.

Consequently, $x \in \Gamma_M(\lambda)$.

Hence $\Gamma_M \subset \Gamma_M(\lambda)$. (1.2)

From (1.1) and (1.2) we infer that $\Gamma_M(\lambda) = \Gamma_M$.

On the other hand, suppose that $\Gamma_M(\lambda) = \Gamma_M$.

If λ was not analytic then for each

positive integer k, there is an n_k such that

$|\lambda_{n_k}|^{1/n_k} > k$ (1.3)

Define $x = \{x_n\}$ by

$$M\left(\frac{|x_n|^{1/n}}{\rho}\right) = \begin{cases} \frac{1}{k}, & \text{if } n = n_k \ (k = 1, 2, \ldots); \\ 0, & \text{otherwise} \end{cases}$$

Then $x \in \Gamma_M$ from (1.3) and $M\left(\frac{|\lambda_n x_n|^{1/n}}{\rho}\right) =$

$$M\left(\frac{|\lambda_{n_k} x_{n_k}|^{1/n_k}}{\rho}\right) > 1$$

Showing that $\lambda_k \notin \Gamma_M$. This is a contradiction to $\Gamma_M(\lambda) = \Gamma_M$.

This contradiction shows that $\lambda \in \wedge$.

This completes the proof.

Proposition 2: Let $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ be two arbitrary fixed sequences of non-zero complex numbers. Then $\Gamma_M(\lambda) \subset \Gamma_M(\mu)$ if and only if

$$\left\{\min\left\{\left|\frac{\mu_k}{\lambda_k}\right|^{1/k}, |\mu_k|^{1/k}\right\}\right\} \text{ is analytic.} \quad (2.1)$$

Proof:

Let A denote the set of those positive integers k for which $|\lambda_k|^{1/k} > 1$.

Let B denote the set of those positive integers k for which $|\lambda_k|^{1/k} \leq 1$.

$$k \in A \Rightarrow \min\left\{\left|\frac{\mu_k}{\lambda_k}\right|^{1/k}, |\mu_k|^{1/k}\right\} = \left|\frac{\mu_k}{\lambda_k}\right|^{1/k}$$

$$k \in B \Rightarrow \min\left\{\left|\frac{\mu_k}{\lambda_k}\right|^{1/k}, |\mu_k|^{1/k}\right\} = |\mu_k|^{1/k}.$$

Hence (2.1) is equivalent to the assertion that $\left\{\left|\frac{\mu_k}{\lambda_k}\right|^{1/k}\right\}$ is analytic for $k \in A$ and

$\left\{|\mu_k|^{1/k}\right\}$ is analytic for $k \in B$. Suppose that this holds and that $x \in \Gamma_M(\lambda)$.

If $k \in A$, write $M\left(\frac{|x_k \mu_k|^{1/k}}{\rho}\right) = M\left(\left(\frac{|x_k \lambda_k|^{1/k}}{\rho}\right)\left(\left|\frac{\mu_k}{\lambda_k}\right|^{1/k}\right)\right)$

If $k \in B$, write $M\left(\frac{|x_k \mu_k|^{1/k}}{\rho}\right) = M\left(\frac{|x_k|^{1/k}}{\rho}\right) |\mu_k|^{1/k}$.

In either case, $M\left(\frac{|x_k \mu_k|^{1/k}}{\rho}\right)$ is arbitrarily small for sufficiently large k. Hence $x \in \Gamma_M(\mu)$.

Thus $\Gamma_M(\lambda) \subset \Gamma_M(\mu)$.

On the other hand, if (2.1) is false, we can find an increasing sequence of positive integers $\{k_r\}$ such that (2.2)

$$\left|\frac{\mu_{k_r}}{\lambda_{k_r}}\right|^{1/k_r} \geq r$$

and $|\mu_{k_r}|^{1/k_r} \geq r$ for $r = 1, 2, 3, \ldots$ (2.3)

If $\left|\lambda_{k_r}\right|^{1/k_r} > 1$ choose $M\left(\frac{\left|x_{k_r}\right|^{1/k_r}}{\rho}\right) = \frac{1}{\left|\lambda_{k_r}\right|^{k_r}}$. Then (2.2)

gives $M\left(\frac{\left|x_{k_r}\mu_{k_r}\right|^{1/k_r}}{\rho}\right) \geq 1$.

If $\left|\lambda_{k_r}\right|^{1/k_r} < 1$ choose $M\left(\frac{\left|x_{k_r}\right|^{1/k_r}}{\rho}\right) = \frac{1}{r}$.

Then (2.3) gives $M\left(\frac{\left|x_{k_r}\mu_{k_r}\right|^{1/k_r}}{\rho}\right) \geq 1$.

Thus in either case $x \in \Gamma_M(\lambda)$ but $x \notin \Gamma_M(\mu)$.

This contradicts our present hypothesis that $\Gamma_M(\lambda) \subset \Gamma_M(\mu)$.

This completes the proof .

Theorem 1:
$(\Gamma_M(\lambda), d)$ is a complete metric space if and only if

$$\liminf_{k\to\infty} \left|\lambda_k\right|^{1/k} > 0 \tag{3.1}$$

Proof: Suppose that (3.1) holds.

Let $\left\{x^{(n)}\right\}$ be any Cauchy sequence in $\Gamma_M(\lambda)$. Given any $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d\left(x^n, x^m\right) = |\lambda|^{1/k}\ M\left(\frac{\left|x_k^{(n)} - x_k^{(m)}\right|^{1/k}}{\rho}\right) < \varepsilon, \tag{3.2}$$

for all $n, m \geq 0$ and all $k \in$.

Let $L = \inf\left\{ \left|\lambda_k\right|^{1/k} : k = 1,2,3, \ldots \right\}$. Then from 3.2) we get

$$M\left(\frac{\left|x_k^{(n)} - x_k^{(m)}\right|^{1/k}}{\rho}\right) < \frac{\varepsilon}{L} \text{ for all } n, m \geq n_0 \tag{3.3}$$

Hence $\left\{x_k^{(n)} : n = 1,2, \ldots\right\}$ is a Cauchy sequence of complex numbers .

So , $M\left(\frac{\left|x_k^{(n)}\right|^{1/k}}{\rho}\right) \to M\left(\frac{\left|x_k\right|^{1/k}}{\rho}\right)$, $(n \to \infty)$,

for all $k = 1,2,3, \ldots$

Now we show that $x \in \Gamma_M(\lambda)$.

Take $x = \{x_k\}$. Letting $m \to \infty$ in (3.2) , we have $d\left(x_k^{(n)}, x\right) \to 0,$ as $n \to \infty$.

From (3.3) and the fact that $\left(x_k^{(n_0)}\right) \in \Gamma_M(\lambda)$ for each fixed n_0 we see that

$$\left|\lambda_k\right|^{1/k} \lim_{n\to\infty} M\left(\frac{\left|x_k^{(n)}\right|^{1/k}}{\rho}\right) \leq \left|\lambda_k\right|^{1/k} M\left(\frac{\left|x_k^{(n_0)}\right|^{1/k}}{\rho}\right) + \frac{\varepsilon}{L}$$

That is $|\lambda_k|\ M\left(\frac{\left|x_k\right|^{1/k}}{\rho}\right) \to 0,$ as $k \to \infty$.

So, $x \in \Gamma_M(\lambda)$. Thus $\Gamma_M(\lambda)$ is complete.

Conversely , Suppose that $\Gamma_M(\lambda)$ is complete .

If (3.1) is not true , then $\left\{ \left|\lambda_k\right|^{1/k} \right\}$ contains subsequence $\left\{\lambda_{k_i}\right\}$ which steadily decreases and tends to zero.

Consider the sequence $\left\{\alpha^{(n)}\right\}$, where

$$\alpha_k^{(n)} = \begin{cases} 1, \text{ if } k = k_1, k_2, \ldots, k_n \\ 0, \text{ other wise.} \end{cases}$$

Then $\alpha^{(n)} \in \Gamma_M(\lambda)$ for all $n = 1,2,\ldots$

For $n > m$, we have

$d\left(\alpha^{(m)}, \alpha^{(n)}\right) = \left|\lambda_{k_{n+1}}\right|^{1/k_{n+1}} \to 0$ a $m \to \infty$

Hence $\left\{\alpha^{(n)}\right\}$ is a cauchy sequence in $\Gamma_M(\lambda)$.

If $\lim_{n\to\infty} \alpha^{(n)}$ exists , then $\lim_{n\to\infty} \alpha^{(n)} = \{1,1,1,\ldots\}$

üüüüüüü=üüüüΓüü$_M(\lambda)$ $\Gamma_M(\lambda)$, cease to be complete , a contradiction .

Hence (3.1) must hold whenever $\Gamma_M(\lambda)$ is complete .

This completes the proof .

Notation: $\wedge\left(\frac{1}{\mu}\right) = \left\{ y = \{y_k\} : \left\{\frac{y_k}{\mu_k}\right\} \in \wedge \right\}$

Theorem 2:

The topological dual of $\left[\Gamma_M(\lambda)\right]$ is $\wedge\left(\frac{1}{\mu}\right)$.

In other words $\left[\Gamma_M(\lambda)\right]^* = \wedge\left(\frac{1}{\mu}\right)$

Proof:

Note that $\Gamma_M(\lambda)$ is the set of all those sequences $\{x_k\}$ such that $M\left(\frac{|x_k|^{1/k}}{\rho}\right) \to 0$ and $M\left(\frac{|\lambda_k x_k|^{1/k}}{\rho}\right) \to 0,$ as $k \to \infty$. These two conditions together are equivalent to

$$M\left(\frac{|\mu_k x_k|^{1/k}}{\rho}\right) \to 0, \text{ as } k \to \infty \quad (4.1)$$

where $\mu_k = \text{Max}\left\{1, [\lambda_k]^{1/k}\right\}$

We recall that $\delta^{(k)}$ has 1 in the k^{th} place and zero's elsewhere .

If we take $x = \delta^{(k)}$, then $\left\{M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right\} =$

$$\left\{\frac{M(0)}{\rho}, \frac{M(0)^{1/2}}{\rho}, \ldots, \frac{M(1)^{1/k}}{\rho}, \frac{M(0)^{1/k+1}}{\rho}, \ldots\right\}$$

$$= \left\{0, 0, \ldots, \frac{M(1)^{1/k}}{\rho}, 0, \ldots\right\}$$

which is a null sequence . Hence $\delta^{(k)} \in \Gamma_M(\lambda)$,

$f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \Gamma_M(\lambda)$ and $f \in \left[\Gamma_M(\lambda)\right]^*$, where $\Gamma_M{}^*$ is the dual space of Γ_M . Take $x = \delta^{(k)} \in \Gamma_M(\lambda)$.

Then

$$|\mu_k y_k|\left(\frac{1}{\mu_k}\right) \le \|f\|\, d\left(\delta^k, 0\right) < \infty \; \forall k$$

Thus $(\mu_k y_k)$ is a bounded sequence and hence an analytic sequence .In other words , $y \in \wedge\left(\frac{1}{\mu}\right)$.

There fore $\left[\Gamma_M(\lambda)\right]^* = \wedge\left(\frac{1}{\mu}\right)$.

This completes the proof .

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