International Journal of Natural and Engineering Sciences 6 (3) : 25-29, 2012
ISSN: 1307-1149, E-ISSN: 2146-0086, www.nobel.gen.tr

A Method For Defuzzification Based on Symmetrical Distance and Vicinity

Rahim SANEIFARD
Department of Mathematics, Urmia Branch, Islamic Azad University, Oroumieh, IRAN

***Corresponding Author**
e-mail: srsaneeifard@yahoo.com

Received : February 27, 2012
Accepted : April 02, 2012

Abstract

In this article, the authors discuss the problem of defuzzification by minimizing the weighted distance between two fuzzy quantities. Also, this study obtains the nearest point with respect to a fuzzy number and shows that this point is unique relative to the weighted distance. By utilizing this point, a method is presented for effectively ranking fuzzy numbers and their images to overcome the deficiencies of the previous techniques. Finally, several numerical examples following the procedure indicate the ranking results to be valid.

Keywords: Fuzzy number; Defuzzification; Ranking; Weighted distance, Weighted point.

INTRODUCTION

Representing fuzzy numbers by proper intervals is an interesting and important problem. An interval representation of a fuzzy number may have many useful applications. By using such a representation, it is possible to apply in fuzzy number approaches some results derived in the field of interval number analysis. For example, it may be applied to a comparison of fuzzy numbers by using the order relations defined on the set of interval numbers. Various authors in ()[16,13] have studied the crisp approximation of fuzzy sets. They proposed a rough theoretic definition of that crisp approximation, called the nearest interval approximation of a fuzzy set. Moreover, quite different approach to crisp approximation of fuzzy sets was applied in [3]. They proposed a rough theoretic definition of that crisp approximation, called the nearest ordinary set of a fuzzy set, and they suggested a construction of such a set. They discussed rather discrete fuzzy sets. Their approximation of the given fuzzy set not unique. Thus this article will not discuss this method. Having reviewed the previous interval approximations, this article proposes here a method to find the weighted interval approximation of a fuzzy number that it is fulfills two conditions. In the first, this interval is a continuous interval approximation operator. In the second, the parametric distance between this interval and the approximated number is minimal. In continuance, by using this interval we obtain the nearest weighted point approximation respect to weighted distance and show that this point is unique. The main purpose of this article is that this nearest weighted point can be used as a crisp approximation of a fuzzy number, therefore by means of this approximation this article aims to present a new method for ranking of fuzzy numbers. The paper is organized as follows: In Section 2, this article recalls some fundamental results on fuzzy numbers. In Section 3, a crisp set approximation of a fuzzy number is obtained. In this Section some remarks are proposed and illustrated. In Section 4, a crisp approximation of a fuzzy number is obtained. The proposed method for ranking fuzzy numbers is in Section 5. Discussion and comparison of this work and other methods are carried out in Section 6.

Preliminaries

The basic definitions of a fuzzy number are given in [8,10,14,15] as follows:

Definition 2.1. A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$ that $0 \le r \le 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
2. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
3. $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

Definition 2.2. The trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$, with two defuzzifier x_0, y_0 and left fuzziness $\sigma > 0$ and right fuzziness $\beta > 0$ is a fuzzy set where the membership function is as

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & when \ x_0 - \sigma \le x \le x_0, \\ 1 & when \ x_0 \le x \le y_0, \\ \frac{1}{\beta}(y_0 - x + \beta) & when \ y_0 \le x \le y_0 + \beta, \\ 0 & otherwise. \end{cases}$$

If $x_0 = y_0$, then $u = (x_0, \sigma, \beta)$ is called trapezoidal fuzzy number. The parametric form of triangular fuzzy number is $\underline{u}(r) = x_0 - \sigma + \sigma r$, $\overline{u}(r) = x_0 + \beta - \beta r$.

Definition 2.3. For arbitrary fuzzy number $u \in F$ (F denotes the space of fuzzy numbers) and $0 \le \delta \le 1$, function $f : F \times [\delta, 1] \to F$ such that $f(u, \delta) = (\underline{u}_\delta, \overline{u}_\delta)$ is called delta- vicinity of the fuzzy number u. Then there is

$$\underline{u}_\delta(r) = \begin{cases} \underline{u}(r) & when \ r \in (\delta, 1], \\ \underline{u}(\delta) & when \ r \in [0, \delta], \end{cases} \quad \text{and}$$

$$\overline{u}_\delta(r) = \begin{cases} \overline{u}(r) & when \ r \in (\delta, 1], \\ \overline{u}(\delta) & when \ r \in [0, \delta], \end{cases}$$

If $u = (x_0, y_0, \sigma, \beta)$ is a trapezoidal fuzzy number, the parametric form of it is $u_\delta = (\underline{u}_\delta, \overline{u}_\delta)$ that is as follows:

$$\underline{u}_\delta(r) = \begin{cases} x_0 - \sigma + \sigma r, & when \ r \in (\delta, 1], \\ x_0 - \sigma + \sigma\delta & when \ r \in [0, \delta], \end{cases} \quad \text{and}$$

$$\overline{u}_\delta(r) = \begin{cases} y_0 + \sigma - \beta r, & when \ r \in (\delta, 1], \\ y_0 + \beta + \beta\delta & when \ r \in [0, \delta]. \end{cases}$$

Definition 2.5. [17]. For two arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$, the quantity

$$d_w(u,v) = \left(\int_0^1 f(r)[\underline{u}(r) - \underline{v}(r)]^2 dr + \int_0^1 f(r)[\overline{u}(r) - \overline{v}(r)]^2 dr \right)^{\frac{1}{2}}, \quad (2.2)$$

were $f : [0,1] \to [0,1]$ is a bi-symmetrical (regular) weighted function, is called the bi-symmetrical (regular) weighted distance between u and v based on F.

Definition 2.6. [13]. An operator $I : F \to (Set\ of\ Closed\ Intervals\ in\ \Re)$ is called an interval approximation operator if for any $u \in F$

$(a')\ I(u) \subseteq Supp(u)$,

$(b')\ Core(u) \subseteq I(u)$,

$(c')\ \forall(\varepsilon > 0) \exists(\delta > 0)\ s.t\ d(u,v) < \delta \Rightarrow d(I(u), I(v)) < \varepsilon$

where $d : F \to [0, \infty[$ denotes a metric defined in the family of all fuzzy numbers.

Definition 2.7. [13] An interval approximation operator satisfying in condition (c') for any $u, v \in F$ is called the continuous interval approximation operator.

Nearest Weighted Interval of a Fuzzy Number

Various authors in [13,16] have studied the crisp approximation of fuzzy sets. They proposed a rough theoretic definition of that crisp approximation, called the nearest ordinary set and nearest interval approximation of a fuzzy set. In this section, the researchers will propose another approximation called the weighted interval-value approximation. Let $u = (\underline{u}, \overline{u})$ be an arbitrary fuzzy number. This article will try to find a closed interval $I_{d_w}(u) = [I_L, I_R]$, which is the weighted interval to u with respect to metric d_w. So, this article has to minimize

$$d_w(u, I_{d_w}(u)) = \left(\int_0^1 f(r)\left([\underline{u}(r) - I_L]^2 + [\overline{u}(r) - I_R]^2\right) dr \right)^{\frac{1}{2}}, \quad (3.4)$$

with respect to I_L and I_R. In order to minimize d_w it suffices to minimize

$$\overline{D}_w(I_L, I_R) = d_w^2(I_L, I_R).$$

It is clear that, the parameters I_L and I_R which minimize Eq. (3.4) must satisfy in

$$\nabla \overline{D}_w(I_L, I_R) = \left(\frac{\partial \overline{D}_w}{\partial I_L}, \frac{\partial \overline{D}_w}{\partial I_R} \right) = 0$$

Therefore, this article has the following equations:

$$\begin{cases} \frac{\partial \overline{D}_w(I_L, I_R)}{\partial I_L} = -2\int_0^1 f(r)(\underline{u}(r) - I_L)dr = 0, \\ \frac{\partial \overline{D}_w(I_L, I_R)}{\partial I_R} = -2\int_0^1 f(r)(\overline{u}(r) - I_R)dr = 0. \end{cases} \quad (3.5)$$

The parameters I_L associated with the left bound and I_R associated with the right bound of the nearest weighted interval can be found by using Eq. (3.5) as follows:

$$\begin{cases} I_L = \int_0^1 f(r)\underline{u}(r)dr, \\ I_L = \int_0^1 f(r)\overline{u}(r)dr. \end{cases} \quad (3.6)$$

Remark 3.1 . Since

$$\det\begin{bmatrix} \frac{\partial^2 \overline{D}_w(I_L,I_R)}{\partial I_L^2} & \frac{\partial^2 \overline{D}_w(I_L,I_R)}{\partial I_R \partial I_L} \\ \frac{\partial^2 \overline{D}_w(I_L,I_R)}{\partial I_l \partial I_R} & \frac{\partial^2 \overline{D}_w(I_L,I_R)}{\partial I_R^2} \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 > 0,$$

and

$$\frac{\partial \overline{D}_w^2(I_L,I_R)}{\partial I_L^2} = 1 > 0$$

therefore I_L and I_R given by (3.6), minimize $d_w(u, I_{d_w}(u)$ Therefore, the interval

$$I_{d_w}(u) = \left[\int_0^1 f(r)\underline{u}(r)d\,r, \int_0^1 f(r)\overline{u}(r)d\,r\right] \quad (3.7)$$

is the nearest weighted interval approximation of fuzzy number u with respect to d_w. Now, suppose that this article wants to approximate a fuzzy number by a crisp interval. Thus the researchers have to use an operator I_{d_w} which transforms fuzzy numbers into family of closed intervals on the real line.

Lemma 3.1. [16]. $\left(2\int_0^1 f(r)g(r)d\,r\right)^2 \le 2\int_0^1 f(r)g^2(r)d\,r$

Theorem 3.1. [16]. The operator I_{d_w} is an interval approximation operator, i.e. I_{d_w} is a continuous interval approximation operator.

Nearest Weighted Point of a Fuzzy Number

Let $u = (\underline{u}, u)$ be an arbitrary fuzzy number and $I_{d_w}(u) = [I_L, I_R]$, be the nearest weighted interval of it. If $I_L = I_R$, then $I_{d_w}(u) = [I_L, I_R] = I_w(u)$, is the nearest weighted point approximation to fuzzy number u, and its unique. $I_w(u)$ value is as follows:

The above equation introduces in the following Theorem.

Theorem 4.1. [17]. Let $u = (\underline{u}, \overline{u})$ be a fuzzy number and $f(r)$ be a weighted function. Then $I_w(u)$ is nearest weighted point to fuzzy number u .

Theorem 4.2. [17]. The nearest weighted point approximation to a given fuzzy number u is unique.

Remark 4.1. [17].Let u and v be two fuzzy numbers and λ and μ be positive numbers. Then we have

Remark 4.2. If $u = (x_0, y_0, \sigma, \beta)$ be a trapezoidal fuzzy number, then the nearest weighted point to it is

$$I_w(u) = \frac{x_0 + y_0 + \beta + \sigma}{2} + (\beta - \sigma)\int_0^1 f(r) r\, d\,r.$$

Ordering of Fuzzy Numbers by the Nearest Weighted Point

In this section, the researchers will propose the ranking of fuzzy numbers associated with the nearest weighted point approximation. Ever, the nearest weighted point can be used as a crisp approximation of a fuzzy number, therefore the resulting approximation is used to rank the fuzzy numbers. Thus, $I_w(.)$ is used to rank fuzzy numbers.

Definition 5.1. Let u and $v \in F$ are two fuzzy numbers, and $I_w(u)$ and $I_w(v)$ are the nearest weighted point of them. Define the ranking of u and v by I_w on F , i.e.

(1) $I_w(u) < I_w(v)$ if only if $u \prec v$,

(2) $I_w(u) > I_w(v)$ if only if $u \succ v$,

(3) $I_w(u) = I_w(v)$

if $I_w(u_\delta) < I_w(v_\delta)$ then $u \prec v$,

else if $I_w(u_\delta) > I_w(v_\delta)$ then $u \succ v$,

else $u \sim v$.

Via Theorem 4.1, the nearest weighted point approximation of the δ - vicinity u is as follows:

$$I_w(u) = \int_0^1 f(r)\left(\underline{u}(r) + \overline{u}(r)\right)dr$$

Then, this article formulates the order $\succeq$ and $\preceq$ as $u \succeq v$ if and only if $u \succ v$ or $u \sim v$, $u \preceq v$ if and only if $u \prec v$ or $u \sim v$. This article considers the following reasonable axioms that Wang and Kerre [20] proposed for fuzzy quantities ranking.

Let I be an ordering method, S the set of fuzzy quantities for which the method I can be applied, and A a finite subset of S . The statement two elements u and v in A satisfy that u has a higher ranking than v when I is applied to the fuzzy quantities in A will be written as $u \succ v$ by I on A . $u \sim v$ by I on A , and $u \succeq v$ by I on A are similarly interpreted. The following proposition shows the reasonable properties of the ordering approach I . Let S be the set of fuzzy quantities for which the nearest point method can be applied, and A and A' are two arbitrary finite subsets of S . The following axioms hold.

A_1 **.** For $u \in A$, $u \preceq u$ by I on A .

A_2 **.** For $(u, v) \in A^2$, $u \preceq v$ and $v \preceq u$ by I on A , we should have $u \sim v$.

A_3 **.** For $(u, v, w) \in A^3$, $u \preceq v$ and $v \preceq w$ by I on A , we should have $u \preceq w$.

A_4 **.** For $(u, v) \in A^2$, $\inf Supp(v) > \sup Supp(u)$, we should have $u \preceq v$.

A_5 **.** Let u , v , $u + w$ and $v + w$ be elements of S . If $u \preceq v$ by I on $\{u, v\}$, then $u + w \preceq v + w$.

A_6 **.** Let u , v , $u + w$ and $v + w$ be elements of S . If $u \prec v$ by I on $\{u, v\}$, then $u + w \prec v + w$.

Remark 5.1. Ranking order I_w has the axioms $A_1, A_2, \ldots, A_6$

Proof. The proof is similar to [17].

Remark 5.2. If $u \prec v$, then $-u \succ -v$.

Hence, this article can infer ranking order of the images of the fuzzy numbers.

Examples

In this section, we want compare proposed method with others in [1,2,5,6,7,9,11,12,18,19].

Example 6.1. First of all, this study validates their proposed method with representative examples of [4] with some advantages.

Consider the data used in [10], i.e. the three fuzzy numbers, $A=(6,1,1)$, $B=(6,0.1,1)$, and $C=(6,0,1)$. According to Eq. (4.10), the ranking index values are obtained i.e. $I_w(A)=6$ Accordingly, the ranking order of fuzzy numbers is $C \succ B \succ A$. However, by Chu and Tsao's approach [9], the ranking order is $B \succ C \succ A$. Meanwhile, using CV index proposed [5], the ranking order is $B \succ C \succ A$. It easy to see that the ranking results obtained by the existing approaches [5,9] are unreasonable and are not consistent with human intuition. On the other hand in [10], the ranking result is $C \succ B \succ A$, which is the same as the one obtained by our approach. However, our approach is simpler in the computation procedure. Based on the analysis results from [10], the ranking results using our approach and other approaches are listed in Table (6.1).

Example 6.2. Consider the following sets: $A=(2,1,3)$, $B=(3,3,1)$ and $C=(2.5,0.5,0.5)$. By using this new approach, $I_w(A)=I_w(B)=2.7$ and $I_w(C)=2.5$ .Since $I_w(A)=I_w(B)$, we have to compare $I_w(A_\delta)$ with $I_w(B_\delta)$, where

$$I_w(A_\delta)=8\delta^2-\frac{5}{3}\delta^3-\frac{10}{3} \quad \text{and} \quad I_w(B_\delta)=4\delta^2-\frac{5}{3}\delta^3-\frac{5}{3}$$

Then, $I_w(A_\delta)<I_w(B_\delta)$ for all $\delta\in[0,1]$. Therefore, the ranking is $C \prec A \prec B$ .To compare with some of the other methods in [9], the readers can refer to Table (6.2). Furthermore, to aforesaid example,

$$I_w(-A_\delta)=-8\delta^2+\frac{5}{3}\delta^3+\frac{10}{3}-2.33 \quad ,$$

$$I_w(-B_\delta)=-4\delta^2-\frac{5}{3}\delta^3+\frac{5}{3} \quad \text{and}$$

$I_w(-C)=-2.5$, consequently the ranking order of the images of three fuzzy number is $-B \prec -A \prec -C$ for all $\delta\in[0,1]$. Clearly, this proposed method has consistency in ranking fuzzy numbers and their images, which could not be guaranteed by CV-index method. It easy to see that neither of them is consistent with human intuition.

All the above examples show that this method is more consistent with institution than the previous ranking methods.

CONCLUSION

In this paper, the authors proposed a defuzzification using minimizer of the weighted distance between two fuzzy numbers and by using this defuzzification we proposed a method for ranking of fuzzy numbers. Roughly, there not much difference in our method and theirs. The method can effectively rank various fuzzy numbers and their images.

Table (6.1). Comparative results of Example (6.1)

Fn	New approach	Sign Distance p=1	Sign Distance p=2	Chu-Tsao	Cheng	CV index
A	6.00	6.12	8.52	3.000	6.021	0.028
B	6.15	12.45	8.82	3.126	6.349	0.009
C	6.16	12.50	8.85	3.085	6.351	0.008
Results	$C \succ B \succ A$	$C \succ B \succ A$	$C \succ B \succ A$	$B \succ C \succ A$	$C \succ B \succ A$	$B \succ C \succ A$

Table (6.2). Comparative results of Example (6.2)

Fn	Sign Distance p=2	Distance Minimization	Chu and Tsao	CV index	Magnitude method
A	3.9157	2.5	0.74	0.32	2.16
B	3.9157	2.5	0.74	0.36	2.83
C	3.5590	2.5	0.75	0.08	2.50
Results	$C \prec A \sim B$	$C \sim A \sim B$	$A \sim B \prec C$	$B \prec A \prec C$	$A \prec C \prec B$

REFERENCES

[1] Baldwin JF, Guild NCF. 1979. Comparison fuzzy numbers on the same decision space. Fuzzy Sets and Systems. 2:213-233.

[2] Bass SM, Kwakernaak H. 1977. Rating and ranking of multiple aspect alternatives using fuzzy sets. Automatica. 13:47-58.

[3] Chakrabarty K, Biswas S, Nanda S. 1998. Nearest ordinary set of a fuzzy set: a rough theoretic construction. Polish Academic Science. 46:105-114.

[4] Chang W. 1981. Ranking of fuzzy utilities with triangular membership function. Proceeding of the International conference on policy analysis information system. 105:263-272.

[5] Cheng CH. 1999. Ranking alternatives with fuzzy weights using maximizing set and minimizing set. Fuzzy Sets and System. 105:365-375.

[6] Chen SH. 1985. Ranking fuzzy numbers with maximizing set and minimizing set. Fuzzy Sets and Systems. 17:113-129.

[7] Choobineh F, Li H. 1993. An index for ordering fuzzy numbers. Fuzzy Sets and Systems. 54:287-294.

[8] Chu H, Lee-Kwang H. 1994. Ranking fuzzy values with satisfaction function. Fuzzy Sets and Systems. 64:295-311.

[9] Chu T, Tsao C. 2002. Ranking fuzzy numbers with an area between the centroid point and original point. Computers and Mathematics with Applications. 43:11-117.

[10] Dubois D, Prade H. 1987. The mean value of a fuzzy number. Fuzzy Sets and Systems. 24:279-300.

[11] Ezzati R, Saneifard R. 2010. A new approach for ranking of fuzzy numbers with continuous weighted quasi-arithmetic means. Mathematical Sciences. 4:143-158.

[12] Ezzati R, Saneifard R. 2010. Defuzzification trough a novel approach. Proc.10th Iranian Conference on Fuzzy Systems. 343–348.

[13] Grzegorzewski P. 2002. Nearest interval approximation of a fuzzy number. Fuzzy Sets and Systems. 130:321-330.

[14] Heilpern S. 1992. The expected value of a fuzzy number. Fuzzy Sets and Systems. 47:81-86.

[15] Kauffman A, Gupta MM. 1991. Introduction to Fuzzy Arithmetic: Theory and Application. Van Nostrand Reinhold, New York.

[16] Saneifard, R. 2009. A method for defuzzification by weighted distance. International Journal of Industrial Mathematics. 3:209-217.

[17] Saneifard R, Ezzati R. 2010. Defuzzification through a bi-symmetrical weighted function. Australian Journal of Basic and Application Sciences. 4:4856–4865.

[18] Saneifard R. 2009. Ranking L-R fuzzy numbers with weighted averaging based on level. International Journal of Industrial Mathematics. 2:163-173.

[19] Saneifard R, Allahviranloo T, Hosseinzadeh F, Mikaeilvand N. 2007. Euclidean ranking DMUs with fuzzy data in DEA. Applied Mathematical Sciences. 60:2989-2998.

[20] Wang X, Kerre EE. 2001. Reasonable properties for the ordering of fuzzy quantities (I). Fuzzy Sets and Systems. 118:378-405.

[21] Yager RR, Filev DP. 1993. On the issue of defuzzification and selection based on a fuzzy set. Fuzzy Sets and Systems. 55:255-272.