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On Expansion in Eigenfunctions for Schrödinger Equation with a General Boundary Condition on Finite Time Scale

Nihal YOKUŞ[1] Esra Kır ARPAT[2]
[1]Ankara University., Department of Mathematics, Tandoğan- Ankara, TURKEY
[2]Gazi University., Department of Mathematics, Faculty of Arts and Sciences, Teknikokullar, Ankara, TURKEY

***Corresponding Author**
e-mail: nyokus20@hotmail.com

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Abstract

In this paper we consider the operator L generated in $L^2{}_\nabla(a,b]$ by the boundary problem

$$-[y^\Delta(t)]^\nabla + [q(t) + 2\lambda p(t) - \lambda^2]y(t) = 0,\ t \in (a,b],$$
$$y(a) - hy^\Delta(a) = 0, \quad y(b) + Hy^\Delta(b) = 0$$

where $p(t)$ is continuous, $q(t)$ is partial continuous, $q(t) \geq 0,\ h \geq 0,\ H \geq 0$. We have obtained eigenvalues and eigenfunctions of Schrödinger Operator with a general boundary condition on finite time scale and the formula of convergent expansions in terms of the eigenfunctions in $L^2{}_\nabla(a,b]$ space.

Key words: Time scale, delta derivatives, nabla derivatives, self-adjoint boundary value problem, symetric Green's function.

INTRODUCTION

The first articles on eigenvalues problems for linear Δ-differential equations on time scales have been investigated in [2] and [7].

Guseinov [8] investigated eigenfunction expansions for the simple Sturm-Liouville eigenvalue problem

$$-y^{\Delta\nabla}(t) = \lambda y(t), \quad t \in (a,b) \tag{1}$$

$$y(a) = y(b) = 0 \tag{2}$$

where a and b are some fixed points in a time scale T with a < b and such that the time scale interval (a,b) is not empty.

In that paper $[8]$, existence of the eigenvalues and eigenfunctions for problem (1), (2) is proved and mean square convergent and uniformly convergent expansions in eigenfunctions are established.

Huseynov and Bairamov in [1] have extended the results of [8] to more general following eigenvalue problem

$$-\left[p(t)y^\Delta(t)\right]^\nabla + q(t)y(t) = \lambda y(t), \quad t \in (a,b],$$

$$y(a) - hy^\Delta(a) = 0, \quad y(b) + Hy^\Delta(b) = 0$$

Let us consider the operator L generated in

$$L^2_\nabla(a,b] := \left\{ y : (a,b] \to R \mid \int_a^b y^2(t)\nabla t < \infty \right\}$$

by the eigenvalue problem

$$-\left[y^\Delta(t)\right]^\nabla + \left[q(t) + 2\lambda p(t) - \lambda^2\right]y(t) = 0, \quad t \in (a,b] \tag{3}$$

and the boundary condition

$$y(a) - hy^\Delta(a) = 0, \quad y(b) + Hy^\Delta(b) = 0 \tag{4}$$

We will assume that the following two conditions are satisfied.

(C_1) $p(t)$ is continuous on [a,b] and continuously ∇-differentiable on $(a,b]$, $q(t)$ is piecewise continuous on $[a,b]$, h and H are real numbers.

(C_2) $p(t) > 0$, $q(t) \geq 0$ for $t \in [a,b]$ and $h \geq 0$, $H \geq 0$.

In this paper, the Hilbert-Schmidt theorem on self-adjoint completely continuous operators is applied to show that the eigenvalue problem (3),(4) has a system of eigenfunctions that forms an orthonormal basis for an appropriate Hilbert space. Moreover uniformly convergent expansions in eigenfunctions are obtained when the expanded functions satisfy some smoothness conditions.

Let T be a time scale and $a,b \in \mathrm{T}$ be fixed points with a < b such that the time scale interval

$$(a,b)=\{t\in \mathrm{T} : a<t<b\}$$

is not empty. For standard notions and notations connected to time scales calculus we refer to [5,6].

L^2 – Convergent Expansion

Denote by Ή the Hilbert space of all real ∇-measurable functions y: (a, b]→ R such that $y(b) = 0$ in the case b is left-scattered and H = 0, and that

$$\int_a^b y^2(t)\nabla t<\infty ,$$

with the inner product

$$\langle y,z\rangle=\int_a^b y(t)z(t)\nabla t$$

and the norm

$$\|y\|=\sqrt{\langle y,y\rangle}=\left\{\int_a^b y^2(t)\nabla t\right\}^{\frac{1}{2}}.$$

Next denote by Ɗ the set of all functions $y\in$ Ή satisfying the following three conditions

(i) y is continuous on $[a, \sigma(b)]$, where σ denotes the forward jump operator.

(ii) $y^{\Delta}(t)$ is defined for $t\in[a,b]$ and

$$y(a)-hy^{\Delta}(a)=0,\quad y(b)+Hy^{\Delta}(b)=0$$

(iii) $y^{\Delta}(t)$ is ∇-differentiable on $(a,b]$ and $\left[y^{\Delta}(t)\right]^{\nabla}\in$ Ή.

Obviously Ɗ is a linear subset dense in Ή. Now we define the operator

L : Ɗ ⊂ Ή → Ή as follows. The domain of definition of L is Ɗ and we put

$$(Ly)(t)=-\left[y^{\Delta}(t)\right]^{\nabla}+\left[q(t)+2\lambda p(t)\right]y(t),\quad t\in(a,b],$$

for $y\in$ Ɗ.

Definition 1. $\lambda\in\mathbb{C}$ is called an eigenvalue of problem(3)-(4) if there exists a nonidentically zero function $y\in$ Ɗ such that

$$-\left[y^{\Delta}(t)\right]^{\nabla}+\left[q(t)+2\lambda p(t)-\lambda^2\right]y(t)=0,\quad t\in(a,b].$$

The function y is called an eigenfunction of problem (3)-(4), corresponding to the eigenvalue λ. We see that the eigenvalue problem (3)-(4) is equivalent to the equation

$$Ly-\lambda y=-\left[y^{\Delta}(t)\right]^{\nabla}+[q(t)+2\lambda p(t)-\lambda]y,\ y\in Ɗ,\ y\neq 0 \quad (5)$$

Theorem 1. Under the condition (C_1) we have, for all $y,z\in$ Ɗ,

(i) $\langle Ly,z\rangle=\langle y,Lz\rangle$, (6)

(ii)

$$\langle Ly,y\rangle=h\left[y^{\Delta}(a)\right]^2+H\left[y^{\Delta}(b)\right]^2+\int_a^b\left[y^{\Delta}(t)\right]^2\Delta t+\int_a^b[q(t)+2\lambda p(t)]y^2(t)\nabla t. \quad (7)$$

Proof. We have for all $y,z\in$ Ɗ

(i)

$$\langle Ly,z\rangle=\int_a^b\left\{-\left[y^{\Delta}(t)\right]^{\nabla}+[q(t)+2\lambda p(t)]y(t)\right\}z(t)\nabla t$$

$$=-y^{\Delta}(t)z(t)|_a^b+\int_a^b y^{\Delta}(t)z^{\Delta}(t)\Delta t+\int_a^b[q(t)+2\lambda p(t)]y(t)z(t)\nabla t$$

$$=-y^{\Delta}(t)z(t)|_a^b+y(t)z^{\Delta}(t)|_a^b-\int_a^b y(t)\left[z^{\Delta}(t)\right]^{\nabla}\nabla t+\int_a^b[q(t)+2\lambda p(t)]y(t)z(t)\nabla t$$

$$=\int_a^b y(t)\left\{-\left[z^{\Delta}(t)\right]^{\nabla}+[q(t)+2\lambda p(t)]z(t)\right\}\nabla t$$

$$=\langle y,Lz\rangle$$

where we have used the boundary conditions (4) for functions $y,z\in$ Ɗ.

Simultaneously we have also got

(ii)

$$\langle Ly,y\rangle=-y^{\Delta}(t)y(t)|_a^b+\int_a^b\left[y^{\Delta}(t)\right]^2\Delta t+\int_a^b[q(t)+2\lambda p(t)]y^2(t)\nabla t$$

$$=h\left[y^{\Delta}(a)\right]^2+H\left[y^{\Delta}(b)\right]^2+\int_a^b\left[y^{\Delta}(t)\right]^2\Delta t+\int_a^b[q(t)+2\lambda p(t)]y^2(t)\nabla t$$

(6) shows that the operator L is symmetric(self-adjoint) and (7) shows that under the additional condition (C_2), it is positive

$\langle Ly, y\rangle > 0$ for all $y \in D$, $y \neq 0$.

Therefore all eigenvalues of the operator L are real and positive and any two eigenfunctions corresponding to the distinct eigenvalues are orthogonal.

Now we would like to show that the existence of eigenvalues for problem (3)-(4).

Theorem 2. $\ker L = \{ y \in D : Ly = 0\} = \{0\}$.

Proof. If $y \in D$ and $Ly = 0$, then from (7) we have by the condition (C_2) that $y^{\Delta}(t) = 0$ for $t \in (a,b]$ and hence $y(t)$=constant on $[a,b]$. Then using boundary conditions (4) we get that $y(t) \equiv 0$.

It follows that the inverse operator L^{-1} exists.

Theorem 3. The Green function $G(t,s)$ of (3)-(4) is defined as

$$G(t,s) = \begin{cases} G_1(t,s), & \operatorname{Im}\lambda \le 0 \\ G_2(t,s), & \operatorname{Im}\lambda \ge 0 \end{cases} \tag{8}$$

Furthermore the Green function is symetric that is $G(t,s) = G(s,t)$ for t,s. Where $G_1(t,s)$ on the plane $\operatorname{Im}\lambda \le 0$ is defined as

$$G_1(t,s) = -\frac{1}{w_1}\begin{cases} u_1(t)v_1(s), & t \le s \\ u_1(s)v_1(t), & t \ge s \end{cases}$$

and $G_2(t,s)$ on the plane $\operatorname{Im}\lambda \ge 0$ is defined as

$$G_2(t,s) = -\frac{1}{w_2}\begin{cases} u_2(t)v_2(s), & t \le s \\ u_2(s)v_2(t), & t \ge s \end{cases}$$

In here, $u_1(t)$ and $v_1(t)$ are the solution of (3) satisfying boundary conditions

$$u_1(a) = h, \quad u_1^{\Delta}(a) = 1; \quad v_1(b) = H, \quad v_1^{\Delta}(b) = -1$$

and

$$u_2(a) = h, \quad u_2^{\Delta}(a) = 1; \quad v_2(b) = H, \quad v_2^{\Delta}(b) = -1$$

respectively, and w_1 and w_2 are Wronskian of the solutions u and v which are defined as

$$w_1 = W_t(u_1, v_1) = u_1(t)v_1^{\Delta}(t) - u_1^{\Delta}(t)v_1(t)$$

and

$$w_2 = W_t(u_2, v_2) = u_2(t)v_2^{\Delta}(t) - u_2^{\Delta}(t)v_2(t)$$

Note that $w_1 \neq 0$ and $w_2 \neq 0$.

Then

$$(L^{-1}f)(t) = \int_a^b G(t,s)f(t)\nabla s, \quad \forall f \in H \tag{9}$$

for any $f \in H$ [3,4].

The equations (8) and (9) imply that L^{-1} is completely continuos (or compact) self-adjoint linear operator in the Hilbert space H.

The eigenvalue problem (5) is equivalent (note that $\lambda = 0$ is not an eigenvalue of L) to the eigenvalue problem

$$Bg = \mu g, \quad g \in H, \quad g \neq 0$$

where

$B = L^{-1}$ and $\mu = \dfrac{1}{\lambda}$.

In other words, if λ is an eigenvalue and $y \in D$ is a corresponding eigenfunction for L, then $\mu = \lambda^{-1}$ is an eigenvalue for B with the same corresponding eigenfunction y conversely, if $\mu \neq 0$ is an eigenvalue and $g \in H$ is a corresponding eigenfunction for B, then $g \in D$ and $\lambda = \mu^{-1}$ is an eigenvalue for L with the same eigenfunction g.

Next we use the following well-known Hilbert-Schmidt theorem [1]. For every completely continuos self-adjoint linear operator B in a Hilbert space H there exists an orthonormal system $\{\varphi_k\}$ of eigenvectors corresponding to eigenvalues $\{\mu_k\}$ $(\mu_k \neq 0)$ such that element $f \in H$ can be written uniquely in the form

$$f = \sum_k c_k \varphi_k + \psi,$$

where $\psi \in \ker B$, that is, $B\psi = 0$. Moreover,

$$Bf = \sum_k \mu_k c_k \varphi_k$$

and if the system $\{\varphi_k\}$ is infinite, then

$$\lim \mu_k = 0 \quad (k \to \infty).$$

As a corollary of the Hilbert-Schmidt theorem we have; If B is a completely continuous self-adjoint linear operator in a Hilbert space Ḣ and if $\ker B = \{0\}$, then the eigenvectors of B form an orthogonal basis of Ḣ.

Applying the corollary of the Hilbert-Schmidt theorem to the operator $B = L^{-1}$ and using the above described connection between the eigenvalues and eigenfunctions of L and the eigenvalues and eigenfunctions of B we use the following result in [1].

Theorem 4. Under the conditions (C_1) and (C_2), for the eigenvalue problem (3)-(4) there exists an orthonormal system $\{\varphi_k\}$ of eigenfunctions corresponding to eigenvalues $\{\lambda_k\}$. Each eigenvalue λ_k is positive and simple. The system $\{\varphi_k\}$ forms an orthonormal basis for the Hilbert space Ḣ. Therefore the number of the eigenvalues is equal to $N = \dim$ Ḣ. Any function $f \in$ Ḣ can be expanded in eigenfunctions φ_k in the form

$$f(t) = \sum_{k=1}^{N} c_k \varphi_k(t) \tag{10}$$

where c_k are the Fourier coefficients of f defined by

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t \tag{11}$$

In the case $N = \infty$ the sum in (10) becomes an infinite series and it converges to the function f in metric of the space Ḣ, that is, in mean square metric

$$\lim_{n\to\infty} \int_a^b \left[f(t) - \sum_{k=1}^{n} c_k \varphi_k(t) \right]^2 \nabla t = 0 \tag{12}$$

Note that since

$$\int_a^b \left[f(t) - \sum_{k=1}^{n} c_k \varphi_k(t) \right]^2 \nabla t = \int_a^b f^2(t)\nabla t - \sum_{k=1}^{n} c_k^2 ,$$

we get from (12) the Parseval equality

$$\int_a^b f^2(t)\nabla t = \sum_{k=1}^{N} c_k^2 . \tag{13}$$

Uniformly Convergent Expansion

In this section, if the conditions (C_1) and (C_2) are satisfied, we prove the following result.

Theorem 5. Let $f : [a,b] \to R$ be a function such that it has a Δ-derivative $f^{\Delta}(t)$ everywhere on $[a,b]$, except at a finite number of points $t_1, t_2, \ldots, t_m$ belonging to (a,b), the Δ-derivative being continuous everywhere except at these points, at which f^{Δ} has finite limits from the left and right. Besides assume that f satisfies the boundary conditions

$f(a) - hf^{\Delta}(a) = 0, \quad f(b) + Hf^{\Delta}(b) = 0$.

Then the series

$$\sum_{k=1}^{\infty} c_k \varphi_k(t) \tag{14}$$

where

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t, \tag{15}$$

converges uniformly on $[a,b]$ to the function f .

Proof. Let the function f is Δ-differentiable everywhere on $[a,b]$ and that f^{Δ} is continuous on $[a,b]$.

Consider the functional

$$J(y) = h\left[y^{\Delta}(a)\right]^2 + H\left[y^{\Delta}(b)\right]^2 + \int_a^b \left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b [q(t) + 2\lambda p(t)] y^2(t)\nabla t$$

so that we have $J(y) \geq 0$. Substituting in the functional $J(y)$

$$y = f(t) - \sum_{k=1}^{n} c_k \varphi_k(t) ,$$

where c_k are defined by (15), we obtain

$$J\left(f - \sum_{k=1}^{n} c_k\varphi_k\right) = h\left[f^{\Delta}(a) - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}(a)\right]^2 + H\left[f^{\Delta}(b) - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}(b)\right]^2$$

$$+ \int_a^b \left(f^{\Delta} - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}\right)^2 \Delta t + \int_a^b (q + 2\lambda p)\left(f - \sum_{k=1}^{n} c_k\varphi_k\right)^2 \nabla t$$

$$= h\left(f^{\Delta}(a)\right)^2 + H\left(f^{\Delta}(b)\right)^2$$

$$-2\sum_{k=1}^{n} c_k\left[hf^{\Delta}(a)\varphi_k^{\Delta}(a) + Hf^{\Delta}(b)\varphi_k^{\Delta}(b)\right]$$

$$+\sum_{k,l=1}^{n} c_k c_l\left[h\varphi_k^{\Delta}(a)\varphi_l^{\Delta}(a) + H\varphi_k^{\Delta}(b)\varphi_l^{\Delta}(b)\right] f^2\Delta t$$

$$+\int_a^b f^{\Delta 2}\Delta t + \int_a^b (q+2\lambda p)$$

$$-2\sum_{k=1}^{n} c_k\left(\int_a^b f^{\Delta}\varphi_k^{\Delta}\Delta t + \int_a^b (q+2\lambda p) f\varphi_k\nabla t\right)$$

$$+\sum_{k,l=1}^{n} c_k c_l\left(\int_a^b \varphi_k^{\Delta}\varphi_l^{\Delta}\Delta t + \int_a^b (q+2\lambda p)\varphi_k\varphi_l\nabla t\right). \quad (16)$$

where $\delta_{k,l}$ is the Kronecker symbol and where we have used the boundary conditions (4),

$$\varphi_k(a) - h\varphi_k^{\Delta}(a) = 0, \quad \varphi_k(b) + H\varphi_k^{\Delta}(b) = 0, \quad (17)$$

and the equation

$$-\left[\varphi_k^{\Delta}(t)\right]^{\nabla} + (q+2\lambda p)(t)\varphi_k(t) = \lambda_k\varphi_k(t)$$

Therefore we have from (16)

$$J\left(f - \sum_{k=1}^{n} c_k\varphi_k\right) = h\left[f^{\Delta}(a)\right]^2 + H\left[f^{\Delta}(b)\right]^2 + \int_a^b [f^{\Delta 2} + (q+2\lambda p) f^2]\Delta t - \sum_{k=1}^{n}\lambda_k c_k^2$$

Since the left-hand side is nonnegative we get the inequality

$$\sum_{k=1}^{\infty}\lambda_k c_k^2 \le h\left[f^{\Delta}(a)\right]^2 + H\left[f^{\Delta}(b)\right]^2 + \int_a^b\left[f^{\Delta 2} + (q+2\lambda p) f^2\right]\Delta t \quad (18)$$

analogous to Bessel's inequality, and the convergence of the series on the left follows. All the terms of this series are nonnegative, since $\lambda_k > 0$.

Note that the proof of (18) is entirely unchanged if we assume that the function f satisfies only the conditions stated in the theorem. Indeed, when integrating by parts, it is sufficient to integrate over the intervals on which f^{Δ} is continuous and then add all these integrals (the integrated terms vanish by (4), (17), and the fact that f, φ_k and φ_k^{Δ} are continuous on $[a,b]$).

We now show that the series

$$\sum_{k=1}^{n}\left|c_k\varphi_k(t)\right| \quad (19)$$

is uniformly convergent on the interval $[a,b]$. Obviously from this the uniformly convergence of series (14) will follow.

Using the integral equation

$$\varphi_k(t) = \lambda_k\int_a^b G(t,s)\varphi_k(s)\nabla s$$

which follows from $\varphi_k = \lambda_k L^{-1}\varphi_k$ by (9), we can rewrite (19) as

$$\sum_{k=1}^{n}\lambda_k\left|c_k g_k(t)\right|, \quad (20)$$

where

$$g_k(t) = \int_a^b G(t,s)\varphi_k(s)\nabla s$$

can be regarded as the Fourier coefficient of $G(t,s)$ as a function of s . By using inequality (18), we can write

$$\sum_{k=1}^{\infty}\lambda_k g_k^2(t) \le h\left[G^{\Delta_s}(t,a)\right]^2 + H\left[G^{\Delta_s}(t,b)\right]^2 + \int_a^b\left[G^{\Delta_s 2}(t,s) + (q(s)+2\lambda p(s))G^2(t,s)\right]\Delta s \quad (21)$$

where $G^{\Delta_s}(t,s)$ is the delta derivative of $G(t,s)$ with respect to s . The function appearing under the integral sign is bounded (see (8)) and it follows from (21) that

$$\sum_{k=1}^{\infty}\lambda_k g_k^2(t) \le M,$$

where M is a constant. Now replacing λ_k by $\sqrt{\lambda_k}\sqrt{\lambda_k}$, we apply the Cauchy-Schwarz inequality to the segment of series (20),

$$\sum_{k=m}^{m+p}\lambda_k\left|c_k g_k(t)\right| \le \sqrt{\sum_{k=m}^{m+p}\lambda_k c_k^2}\sqrt{\sum_{k=m}^{m+p}\lambda_k g_k^2(t)} \le \sqrt{M}\sqrt{\sum_{k=m}^{m+p}\lambda_k c_k^2}$$

and this inequality, together with the convergence of the series with terms $\lambda_k c_k^2$ (see(18)), at once implies that series (20), and hence series (19) is uniformly convergent on the interval $[a,b]$. Denote the sum of series (14) by $f_1(t)$

$$f_1(t) = \sum_{k=1}^{\infty} c_k\varphi_k(t) \quad (22)$$

Since the series in (22) is convergent uniformly on $[a,b]$, we can multiply both sides of (22) by $\varphi_l(t)$ and then ∇ integrate it term-by-term to get

$$\int_a^b f_1(t)\varphi_l(t)\nabla t = c_l.$$

Therefore the Fourier coefficients of f_1 and f are the same. Then the Fourier coefficients of the difference $f_1 - f$ are zero and applying the Parseval equality (13) to the function $f_1 - f$ we get that $f_1 - f = 0$, so that the sum of series (14) is equal to $f(t)$.

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